

Deformation Expression for Elements of Algebras (I) –(Jacobi’s theta functions and \ast -exponential functions)–

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This paper presents a preliminary version of the deformation theory of expressions of elements of algebras. The notion of $*$ -functions is given. Several important problems appear in simplified forms, and these give an intuitive bird's-eye of the whole theory, since these problems are main target of this series. These will give a toy model of the philosophical “the theory of observations”.

This series will continue to twelve chapters at least.

1 Definition of $*$ -functions and intertwiners

Let $\mathbb{C}[w]$ be the space of polynomials in one variable w . For a complex parameter τ , we define a new product $*_\tau$ on this space

$$(1.1) \quad f *_\tau g = \sum_{k \geq 0} \frac{\tau^k}{2^k k!} \partial_w^k f \partial_w^k g \quad (= f e^{\frac{\tau}{2} \overleftarrow{\partial}_w \overrightarrow{\partial}_w} g).$$

We easily see that the $*_\tau$ makes $\mathbb{C}[w]$ a commutative associative algebra, which we denote by $(\mathbb{C}[w], *_\tau)$. If $\tau=0$, then $(\mathbb{C}[w], *_0)$ is the usual polynomial algebra, and $\tau \in \mathbb{C}$ is called a

deformation parameter. What is deformed is not the algebraic structure, but the expression of elements.

We see easily that $w *_\tau w = w^2 + \frac{\tau}{2}$, $w *_\tau w *_\tau w = w^3 + \frac{3\tau}{2}w$, etc.

1.1 Intertwiners and infinitesimal intertwiners

It is not hard to verify that the mapping

$$(1.2) \quad e^{\frac{\tau}{4}\partial_w^2} : (\mathbb{C}[w], *_0) \rightarrow (\mathbb{C}[w], *_\tau)$$

gives an algebra isomorphism. That is, $e^{\frac{\tau}{4}\partial_w^2}$ has the inverse mapping $e^{-\frac{\tau}{4}\partial_w^2}$ and the following holds

$$(1.3) \quad e^{\frac{\tau}{4}\partial_w^2}(f *_0 g) = (e^{\frac{\tau}{4}\partial_w^2} f) *_\tau (e^{\frac{\tau}{4}\partial_w^2} g).$$

This isomorphism $I_0^\tau = e^{\frac{\tau}{4}\partial_w^2}$ is called the **intertwiner**. We define also $I_\tau^{\tau'} = I_0^\tau (I_0^\tau)^{-1}$ as the intertwiner from $*_\tau$ onto $*_{\tau'}$, and its differential

$$dI_\tau = I_\tau^{\tau+d\tau} = \frac{d}{d\tau'} I_\tau^{\tau'} \Big|_{\tau'=\tau} = \frac{1}{4} \partial_w^2$$

is called the **infinitesimal intertwiner**.

Intertwiners are powerful tool to compute $*_\tau$ -products. We compute $w_{*\tau}^n$ by $I_0^\tau w^n$:

$$(1.4) \quad w_{*\tau}^n = P_n(w, \tau) = \sum_{k \leq [n/2]} \frac{n!}{4^k k! (n-2k)!} \tau^k w^{n-2k}.$$

For the later use, we note that intertwiners leaves the *parity* invariant. That is,

Proposition 1.1 $w_{*\tau}^{2n}$ is written by a polynomial $p(w^2)$ of w^2 , and $w_{*\tau}^{2n+1}$ is a polynomial written in the form $w p(w^2)$.

Let $Hol(\mathbb{C})$ be the space of all entire functions on \mathbb{C} with the topology of uniform convergence on compact domains. $Hol(\mathbb{C})$ is known to be a Fréchet space defined by a countable family of seminorms. It is easy to see that the product $*_\tau$ extends naturally for $f, g \in Hol(\mathbb{C})$ if f or g is a polynomial. We easily see the following:

Theorem 1.1 For every polynomial $p(w)$, the multiplication operator $p(w) *_\tau$ is a continuous linear mapping of $Hol(\mathbb{C})$ into itself.

By polynomial approximations, the associativity $f *_\tau (g *_\tau h) = (f *_\tau g) *_\tau h$ holds if two of f, g, h are polynomials.

By giving the inductive limit topology, $\mathbb{C}[w]$ is a complete topological algebra, and $Hol(\mathbb{C})$ is a topological $\mathbb{C}[w]$ bi-module.

By Theorem 1.1, every polynomial $p(w)$ may be viewed as a linear operator of $Hol(\mathbb{C})$ into itself. Thus, for instance, one may consider the differential equation

$$\frac{d}{dt}f_t(w) = w_*^\ell * f_t(w), \quad f_0(w) = 1.$$

In ordinary calculus, this is treated by using slightly transcendental elements $\sum_k \frac{t^k}{k!} w_*^{n\ell}$ via the notion of convergence, which is familiar in the ordinary text book of calculus, and there is no problem for the case $\ell = 1$.

1.2 *-exponential functions and τ -expressions

Although the ordinary exponential function e^{aw} is not a polynomial, the intertwiner extends to give a family of elements

$$(1.5) \quad I_0^\tau(e^{2aw}) = e^{2aw+a^2\tau} = e^{a^2\tau} e^{2aw}, \quad \tau \in \mathbb{C}$$

and Taylor expansion formula gives

$$(1.6) \quad e^{2aw} *_\tau e^{2bw} = e^{2(a+b)w+2ab\tau}, \quad e^{2aw} *_\tau f(w) = e^{2aw} f(w+a\tau)$$

for every $f \in Hol(\mathbb{C})$. Using this, we see the associativity

$$(1.7) \quad e^{2aw} *_\tau (e^{2bw} *_\tau f(w)) = (e^{2a\tau} *_\tau e^{2bw}) *_\tau f(w)$$

for every $f \in Hol(\mathbb{C})$. Computation via intertwiners gives

$$I_\tau^{\tau'}(e^{sw}) = e^{\frac{1}{4}(\tau' - \tau)s^2} e^{sw}, \quad I_\tau^{\tau'}(e^{\frac{1}{4}s^2\tau} e^{sw}) = e^{\frac{1}{4}s^2\tau'} e^{sw}.$$

Noting (1.5), we write the family $\{e^{\frac{1}{4}s^2\tau} e^{sw}; \tau \in \mathbb{C}\}$ by e_*^{sw} and call this the ***-exponential function**.

1.2.1 Remarks for the notations

In what follows we use notations such as

$$:e_*^{sw}:_\tau = e^{\frac{1}{4}s^2\tau} e^{sw}$$

and call the r.h.s. the τ -expression of e_*^{sw} . Similarly we denote $:w_*^n:_\tau = P_n(w, \tau)$ and call the r.h.s. the τ -expression of w_*^n .

In general, for a polynomial or an exponential function $f(w)$, there is a family of functions

$$\{f_\tau(w); \tau \in \mathbb{C}\}, \quad f_\tau = I_0^\tau(f).$$

We denote this family by $f_*(w)$ and we view $f_*(w)$ as an *element* of the abstract algebra. We refer to such objects as a ***-functions**. Furthermore, we denote these as

$$(1.8) \quad :f_*(w):_\tau = f_\tau(w)$$

by using the notation $:\bullet:_\tau$. $:f_*:_\tau$ is viewed as the τ -**expression** of f_* . For instance, we write

$$:aw_*+b:_\tau = aw+b, \quad :2w_*^2:_\tau = 2w^2+\tau, \quad :2w_*^3:_\tau = 2w^3+3\tau w, \quad :w_*^n:_\tau = P_n(w, \tau) \text{ (cf.(1.4))}.$$

For $*$ -functions $f_*(w), g_*(w)$, the notations $f_*(w)*g_*(w)$ and $f_*(w)+g_*(w)$ without expression parameter signs mean that these are computed under the same expression parameter. The usual calculations will be done in this manner. Calculations such as

$$:f_*(w):_\tau + :g_*(w):_{\tau'}, \quad :f_*(w):_\tau :g_*(w):_{\tau'}$$

are forbidden, though these are calculations in ordinary functions.

It is sometimes very convenient to regard the infinitesimal intertwiner as a linear connection on the trivial bundle $\coprod_{\tau \in \mathbb{C}} Hol(\mathbb{C})$. But, the parallel transformation do no necessarily defined. If $I_0^\tau f$ is defined on some domain D , this is viewed as a parallel section defined on D . In the later section, we treat multi-valued parallel sections if D is not simply connected.

As the intertwiner I_0^τ is an isomorphism on the space of polynomials, the following is easy to see:

Corollary 1.1 *The unique factorization holds for $*$ -polynomials $p_*(w)$, i.e. $p_*(w)$ is written uniquely in the form*

$$p_*(w) = a_0(w+\lambda_1)^{\ell_1}*(w+\lambda_2)^{\ell_2}*\dots*(w+\lambda_k)^{\ell_k}.$$

By the product formula (1.1) we have the exponential law

$$(1.9) \quad :e_*^{sw}:_\tau *_\tau :e_*^{tw}:_\tau = :e_*^{(s+t)w}:_\tau, \quad \forall \tau \in \mathbb{C}.$$

Furthermore $:e_*^{tw}:_\tau$ is the solution of the differential equation $\frac{d}{dt}g(t) = w*_\tau g(t)$ with the initial condition $g(0)=1$ for every τ . By the uniqueness of solutions, the exponential law with the ordinary exponential function e^s

$$e_*^{tw}e^s = e_*^{tw+s}$$

holds. As $I_0^\tau(e^{tw}) = :e_*^{tw}:_\tau$ and $I_0^\tau(w^n) = :w_*^n:_\tau$, we see that

$$:e_*^{tw}:_\tau = \sum_n \frac{t^n}{n!} :w_*^n:_\tau.$$

Thus, it is natural to view e_*^{sw} as a single element, rather than a parallel section defined on the whole plane \mathbb{C} , or a family of elements which are mutually intertwined.

The exponential law is written as $e_*^{sw} *_\tau e_*^{tw} = e_*^{(s+t)w}$, and the equation itself is written as $\frac{d}{dt}g_*(t) = w*_g(t)$, $g_*(0)=1$ without the subscript τ .

We define naturally

$$\sin_*(w+s) = \frac{1}{2i}(e_*^{i(w+s)} - e_*^{-i(w+s)}), \quad \cos_*(w+s) = \frac{1}{2}(e_*^{i(w+s)} + e_*^{-i(w+s)}).$$

Lemma 1.1 For every $f \in \text{Hol}(\mathbb{C})$, the formula (1.6) gives

$$(1.10) \quad :e_*^{2sw} :_\tau *_\tau f(w) = e^{2sw+s^2\tau} f(w+s\tau)$$

and it holds the following associativity in every τ -expression

$$e_*^{rw} * (e_*^{sw} * f) = e_*^{(r+s)w} * f = e_*^{rw} * (f * e_*^{sw}).$$

As $:e_*^{2nw} :_\tau = e^{n^2\tau} e^{2nw}$, if $\text{Re}\tau < 0$, then we easily see that $\lim_{n \rightarrow \infty} :e_*^{2nw} :_\tau = 0$ uniformly on every compact domain. However, this does not necessarily imply that $\lim_{n \rightarrow \infty} :e_*^{2nw} * f :_\tau = 0$, for (1.10) gives

$$\lim_{n \rightarrow \infty} :e_*^{2nw} :_\tau *_\tau f(w) = \lim_{n \rightarrow \infty} e^{n^2\tau+2nw} f(w+n\tau).$$

Note So far, the generator w is fixed. Note here that a change of generators $w \rightarrow w' = aw$ causes often a confusion, since

$$(1.11) \quad :e_*^{(ta)w} :_\tau = e^{\frac{1}{4}(ta)^2\tau} e^{(ta)w} \neq :e_*^{t(w')} :_\tau = e^{\frac{1}{4}t^2\tau} e^{tw'}.$$

The effect of changing generator systems will be discussed in the forthcoming paper. Here we note that a change of an expression parameter is recovered by the “square” of a change of generators. Thus a change of expression parameter may be viewed as a “square root” of a coordinate transformation.

Remark By definition of $*$ -exponential function, we have $e_*^{0w} = 1$ for every τ -expression. This notation is a little confusing, for it does not imply $:e_*^{aw} :_\tau|_{w=0} = 1$. Instead, it is computed as

$$:e_*^{aw} :_\tau|_{w=0} = e^{\frac{1}{4}a^2\tau} e^{a0} = e^{\frac{1}{4}a^2\tau}.$$

Hence the evaluation $|_{w=0}$ is not a homomorphism:

$$:e_*^{aw} * e_*^{bw} :_\tau|_{w=0} = e^{\frac{1}{4}(a+b)^2\tau} \neq e^{\frac{1}{4}a^2\tau} e^{\frac{1}{4}b^2\tau}.$$

Star-functions of one variable w may be viewed as simplified version of star-functions of a linear function $\langle \mathbf{a}, \mathbf{u} \rangle$ of several variables for a fixed \mathbf{a} .

So far, we treated the differential equation

$$\frac{d}{dt} f_t(w) = w_*^\ell * f_t(w), \quad f_0(w) = 1$$

in the case $\ell = 1$. The solution is given by $\sum_k \frac{t^k}{k!} w_*^k$. The case $\ell = 2$ will be treated in the later section. It will be seen that $e_*^{tw^2}$ behaves 2-valued complex one parameter group with a branching singular point at τ^{-1} .

In general, by noting that $:w_*^{\ell n}:\tau = w_{*\tau}^{\ell n}$, (1.4) gives that

$$\sum_n \frac{t^n}{n!} :w_*^{\ell n}:\tau = \sum_{m,k \geq 0} \frac{(m+2k)!}{n!} \frac{1}{k!m!} t^n \left(\frac{\tau}{4}\right)^k w^m, \quad \ell n = m+2k.$$

Set the r.h.s. by

$$\sum_m A_m(t, \tau) \frac{1}{m!} w^m, \quad A_m(t, \tau) = \sum_k \frac{(m+2k)!}{([\frac{1}{\ell}(m+2k)])!} t^{[\frac{1}{\ell}(m+2k)]} \frac{1}{k!} \left(\frac{\tau}{4}\right)^k$$

where summation runs through all k such that $\ell n = m+2k$. $A_m(t, \tau)$ is a formal power series $A_m(t, \tau) = \sum_k A_{m,k}(t^{\frac{1}{\ell}}) \frac{1}{k!} \left(\frac{\tau}{4}\right)^k$ of τ . By this we have the following:

Proposition 1.2 *If $\ell \geq 3$, then for every $t \neq 0$, the radius of convergence of the power series $\sum_{k=0}^{\infty} A_{m,k}(t^{\frac{1}{\ell}}) \frac{1}{k!} \left(\frac{\tau}{4}\right)^k$ is 0. That is, $e_*^{tw_*^{\ell}}$ cannot be defined as a power series for $\ell \geq 3$.*

Proof Note that the lowest term of $P_{\ell n}(w, \tau)$ is $\frac{(\ell n)!}{4^{\ell n/2}(\ell n/2)!} \tau^{\ell n/2}$ or $\frac{(\ell n)!}{4^{[\ell n/2]}([\ell n/2])!} \tau^{[\ell n/2]} w$. Denote this by a_n . If $\ell \geq 3$, then

$$\lim_{n \rightarrow \infty} \frac{(n+1)|a_n|}{|a_{n+1}|} = 0.$$

This means the radius of convergence of $A_0(t, \tau)$ is 0. Others are similarly. \square

In the later section, $e_*^{tw_*^{\ell}}$, $\ell \geq 3$, is defined as a certain object which has some similarity to a *multi-valued complex one parameter group* with a branching singular point at $t = 0$.

1.3 Applications to generating functions

As in the ordinary calculus, several generating functions are obtained by exponential functions. In this section, we apply these to $*$ -exponential functions to obtain slight generalizations.

1.3.1 The generating function of Hermite polynomials

The generating function of Hermite polynomials is given as follows:

$$e^{\sqrt{2}tx - \frac{1}{2}t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

This is the Taylor expansion formula of $:e_*^{\sqrt{2}tw}:-_1$.

Noting that $e_*^{taw} = \sum \frac{t^n}{n!} (aw)_*^n$, and setting

$$e_*^{\sqrt{2}tw} = \sum_{n \geq 0} (\sqrt{2}w)_*^n \frac{t^n}{n!},$$

we see $H_n(w) = :(\sqrt{2}w)_*^n:_{-1}$. Note that $H_n(w)$ is a polynomial of degree n . For every $\tau \in \mathbb{C}$ we define $*$ -Hermite polynomials $H_n(w, *)$ by

$$(1.12) \quad e_*^{\sqrt{2}tw} = \sum_{n \geq 0} H_n(w, *) \frac{t^n}{n!}, \quad (H_n(w, \tau) = :H(w, *):_\tau, \quad H_n(w, -1) = H_n(w)).$$

Since $\frac{d}{dt}e_*^{\sqrt{2}tw} = \sqrt{2}w_*e_*^{\sqrt{2}tw}$, we have

$$(1.13) \quad \frac{\tau}{\sqrt{2}}H'_n(w, \tau) + \sqrt{2}wH_n(w, \tau) = H_{n+1}(w, \tau)$$

where $H'_n(w, \tau) = \frac{\partial}{\partial w}H(w, \tau) = : \frac{\partial}{\partial w}H(w, *):_\tau$. The exponential law yields

$$\sum_{k+\ell=n} \frac{n!}{k!\ell!} H_k(w, *) * H_\ell(w, *) = H_n(w, *).$$

On the other hand, taking $\frac{\partial}{\partial w}$ of both sides of (1.12) gives

$$\sqrt{2}nH_{n-1}(w, *) = H'_n(w, *).$$

Differentiating (1.13) again and using the above equality gives

$$\tau H''_n(w, \tau) + 2wH'_n(w, \tau) - 2nH_n(w, \tau) = 0.$$

By $\sqrt{2}tw + \frac{\tau}{2}t^2 = \frac{\tau}{2}(t + \frac{\sqrt{2}}{\tau}w)^2 - \frac{1}{\tau}w^2$, the Hermite polynomial $H_n(w, *)$ is obtained by the following formula:

$$\begin{aligned} H_n(w, \tau) &= \frac{d^n}{dt^n} :e_*^{\sqrt{2}tw}:_\tau \Big|_{t=0} = \frac{d^n}{dt^n} e^{\frac{\tau}{2}(t + \frac{\sqrt{2}}{\tau}w)^2 - \frac{1}{\tau}w^2} \Big|_{t=0} \\ &= \frac{d^n}{dt^n} e^{\frac{\tau}{2}(t + \frac{\sqrt{2}}{\tau}w)^2} \Big|_{t=0} e^{-\frac{1}{\tau}w^2} = e^{-\frac{1}{\tau}w^2} \left(\frac{\tau}{\sqrt{2}}\right)^n \frac{d^n}{dw^n} e^{\frac{1}{\tau}w^2} \end{aligned}$$

To obtain the orthogonality of the family $\{H_n(w, \tau)\}_n$, we suppose $\operatorname{Re} \tau < 0$ and we restrict w to the real line. Then, the orthogonality is shown as follows:

$$\int_{\mathbb{R}} e^{\frac{1}{\tau}w^2} H_n(w, \tau) H_m(w, \tau) dw = \int_{\mathbb{R}} \left(\frac{\tau}{\sqrt{2}}\right)^n \frac{d^n}{dw^n} e^{\frac{1}{\tau}w^2} H_m(w, \tau) dw.$$

If $n \neq m$, one may suppose $n > m$ without loss of generality. Hence this vanishes by integration by parts n times. For the case $n = m$, since

$$:e_*^{\sqrt{2}tw}:_\tau = e^{\frac{\tau}{2}t^2 + \sqrt{2}tw} = \sum_{n=0}^{\infty} H_n(w, \tau) \frac{t^n}{n!}$$

by (1.12), we see

$$\frac{1}{n!}H_n(w, \tau) = \sum_{p=0}^{[n/2]} \frac{\sqrt{2}^n \tau^p}{p!(n-2p)!4^p} w^{n-2p}$$

and hence

$$\frac{d^n}{dw^n}H_n(w, \tau) = \sqrt{2}^n n!.$$

It follows

$$\int_{\mathbb{R}} e^{\frac{1}{\tau}w^2} H_n(w, \tau) H_n(w, \tau) dw = n!(-\tau)^n \int_{\mathbb{R}} e^{\frac{1}{\tau}w^2} dw = n!(-\tau)^n \sqrt{-\tau} \sqrt{\pi}.$$

1.3.2 The generating function of Legendre polynomials

The generating function of Legendre polynomials $P_n(z)$ is known to be

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{n=0}^{\infty} P_n(z)t^n, \quad \text{for small } |t|.$$

It is known that $P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2-1)^n$. Hence

$$(1.14) \quad \frac{1}{\sqrt{1-2t(z+a)+t^2}} = \sum_n \frac{1}{2^n n!} \frac{d^n}{da^n} ((z+a)^2-1)^n t^n$$

is viewed as the Taylor expansion of the l.h.s. of (1.14). By the formula of Laplace transform, rewrite the l.h.s. of (1.14), and we see

$$\frac{1}{\sqrt{1-2t(z+a)+t^2}} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{1}{\sqrt{s}} e^{-s(1-2t(z+a)+t^2)} ds = \sum_{n=0}^{\infty} P_n(z+a)t^n.$$

This implies also that

$$(1.15) \quad \left. \frac{d^n}{dt^n} \right|_{t=0} \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{1}{\sqrt{s}} e^{-s(1-2t(z+a)+t^2)} ds = \frac{1}{2^n} \frac{d^n}{da^n} ((z+a)^2-1)^n.$$

Replacing the exponential function in the integrand by the $*$ -exponential function, we define $*$ -Legendre polynomial by

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{1}{\sqrt{s}} e_{*}^{-s(1-2t(w+a)+t^2)} ds = \sum_{n=0}^{\infty} P_n(w+a, *) t^n.$$

As $:e_{*}^{-s(1-2t(w+a)+t^2)}:_{\tau} = e^{\tau s^2 t^2} e^{-s(1-2t(w+a)+t^2)}$, we assume that $\text{Re } \tau < 0$ so that the integral converges.

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{1}{\sqrt{s}} e^{\tau s^2 t^2} e^{-s(1-2t(w+a)+t^2)} ds = \sum_{n=0}^{\infty} P_n(w+a, \tau) t^n, \quad P_n(w+a, \tau) = :P_n(w+a, *):_{\tau}.$$

As the variable z is used formally in (1.15), the same formula as in (1.15) holds in $*$ -exponential functions: That is,

$$\left. \frac{d^n}{dt^n} \right|_{t=0} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{s}} e_*^{-s(1-2t(w+a)+t^2)} ds = \frac{1}{2^n} \frac{d^n}{da^n} ((w+a)_*^2 - 1)_*^n.$$

By this trick we see that

$$(1.16) \quad \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{s}} e_*^{-s(1-2t(w+a)+t^2)} ds = \sum_{n=0}^\infty P_n(w+a, *) t^n = \sum_n \frac{1}{2^n n!} \frac{d^n}{da^n} ((w+a)_*^2 - 1)_*^n t^n.$$

Hence we see that $P_n(w+a, *)$ is a polynomial of degree n and $\frac{d^n}{da^n} P_n(w+a, *) = (2n-1)!!$. This equality is used to obtain the orthogonality of the family $\{P_n(w+a), \tau\}_n$. We restrict $w+a$ to the real line. Integration by parts gives

$$\int_{-1}^1 P_m(w, \tau) P_n(w, \tau) dw = \frac{2}{2n+1} \delta_{m,n}.$$

1.3.3 The generating function of Bessel functions

The generating function of Bessel functions $J_n(z)$ is known to be

$$e^{iz \sin s} = \sum_{n=-\infty}^\infty J_n(z) e^{ins}.$$

Keeping this in mind, we define $*$ -Bessel functions by

$$e_*^{\frac{1}{2}(e^{is}-e^{-is})aw} = \sum_{n=-\infty}^\infty J_n(aw, *) e^{ins}, \quad :J_n(aw, *):_\tau = J_n(aw, \tau), \quad a \in \mathbb{C}.$$

Replacing s by $s + \frac{\pi}{2}$ gives $e_*^{\frac{i}{2}(e^{is}+e^{-is})aw} = \sum_{n=-\infty}^\infty i^n J_n(aw, *) e^{ins}$ and basic symmetric properties hold:

First we see $J_n(aw, *) = (-1)^n J_{-n}(aw, *)$. Replacing w by $-w$ in the first equality gives $J_n(-aw, *) = J_{-n}(aw, *)$. Since

$$:e_*^{\frac{1}{2}(e^{is}-e^{-is})aw}:_\tau = e^{-\frac{a^2}{16}\tau(e^{is}-e^{-is})^2} e^{\frac{1}{2}(e^{is}-e^{-is})aw} = e^{-\frac{a^2}{8}\tau} e^{\frac{a^2}{16}\tau(e^{2is}+e^{-2is})} e^{\frac{1}{2}(e^{is}-e^{-is})aw},$$

$J_n(aw, \tau)$ and $J_n(aw)$ are related by

$$\sum_{n=-\infty}^\infty J_n(aw, \tau) e^{ins} = e^{-\frac{a^2}{8}\tau} e^{\frac{a^2}{16}\tau(e^{2is}+e^{-2is})} \sum_{n=-\infty}^\infty J_n(aw) e^{ins}.$$

Setting $s = 0$, we see in particular

$$1 = \sum_{n=-\infty}^\infty J_n(aw, \tau) = \sum_{n=-\infty}^\infty J_n(aw).$$

The exponential law of l.h.s. of the defining equality gives that

$$e_*^{\frac{1}{2}(e^{is}-e^{-is})aw} * e_*^{\frac{1}{2}(e^{is}-e^{-is})bw} = e_*^{\frac{1}{2}(e^{is}-e^{-is})(a+b)w} = \sum_n J_n(aw+bw, *) e^{ins}.$$

Hence,

$$J_n(aw+bw, *) = \sum_{m=-\infty}^{\infty} J_m(aw, *) * J_{n-m}(bw).$$

If $a^2+b^2 = 1$, then

$$e_*^{\frac{1}{2}(e^{is}-e^{-is})aw} * e_*^{\frac{i}{2}(e^{is}+e^{-is})bw} = e_*^{\frac{1}{2}((a+ib)e^{is}-(a-ib)e^{-is})w} = \sum_n J_n(w, *) (a+ib)^n e^{nis}.$$

$$\sum_{k=-\infty}^{\infty} J_k(aw, *) e^{iks} * \sum_{\ell=-\infty}^{\infty} i^\ell J_\ell(bw, *) e^{i\ell s} = \sum_n J_n(w, *) (a+ib)^n e^{nis}.$$

The generating functions of Bernoulli numbers, Euler numbers and Laguerre polynomials will be treated later sections, for there are some other problems for the treatment.

1.4 Jacobi's theta functions and Imaginary transformations

For arbitrary $a \in \mathbb{C}$, consider the $*$ -exponential function $e_*^{t(w+a)}$. Since

$$:e_*^{t(w+a)}:_\tau = e^{\frac{\tau}{4}t^2} e^{t(w+a)},$$

by assuming $\text{Re } \tau < 0$, this is rapidly decreasing on \mathbb{R} . Hence, we see that both

$$\int_{-\infty}^{\infty} :e_*^{t(w+a)}:_\tau dt, \quad \sum_{n=-\infty}^{\infty} :e_*^{n(w+a)}:_\tau$$

converge absolutely on every compact domain in w to give entire functions of w .

In this section, we treat first a special case $\theta(w, *) = \sum_n e_*^{2inw}$ under the condition $\text{Re } \tau > 0$. If we set $q = e^{-\tau}$, the τ -expression $\theta(w, \tau) = : \theta(w, *) :_\tau$ is given by

$$\theta(w, \tau) = \sum_{n \in \mathbb{Z}} q^{n^2} e^{2niw}.$$

This is Jacobi's elliptic θ -function $\theta_3(w, \tau)$.

Furthermore, Jacobi's elliptic theta functions $\theta_i, i=1, 2, 3, 4$ are all τ -expressions of bilateral geometric series as follows (cf. [AAR]).

$$(1.17) \quad \begin{aligned} \theta_1(w, *) &= \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n e_*^{(2n+1)iw}, & \theta_2(w, *) &= \sum_{n=-\infty}^{\infty} e_*^{(2n+1)iw}, \\ \theta_3(w, *) &= \sum_{n=-\infty}^{\infty} e_*^{2niw}, & \theta_4(w, *) &= \sum_{n=-\infty}^{\infty} (-1)^n e_*^{2niw} \end{aligned}$$

This fact was mentioned first in [O], but no further investigation of this fact has been done.

By the exponential law $e_*^{aw+s} = e_*^{aw}e^s$ for $s \in \mathbb{C}$, which is proved easily, we see that $\theta_i(w, *)$ is 2π -periodic for every i . (Precisely, $\theta_1(w, *)$, $\theta_2(w, *)$ are alternating π -periodic, and $\theta_3(w, *)$, $\theta_4(w, *)$ are π -periodic.) Furthermore the exponential law $e_*^{aw+bw} = e_*^{aw} * e_*^{bw}$ of (1.9) gives the trivial identities

$$e_*^{2iw} * \theta_i(w, *) = \theta_i(w, *), \quad (i=2, 3), \quad e_*^{2iw} * \theta_i(w, *) = -\theta_i(w, *), \quad (i=1, 4).$$

For every τ such that $\operatorname{Re} \tau > 0$, τ -expressions of these are given by using $:e_*^{2iw}:_\tau = e^{-\tau}e^{2iw}$ and (1.10) as follows:

$$(1.18) \quad \begin{aligned} e^{2iw-\tau} \theta_i(w+i\tau, \tau) &= \theta_i(w, \tau), \quad (i=2, 3), \\ e^{2iw-\tau} \theta_i(w+i\tau, \tau) &= -\theta_i(w, \tau), \quad (i=1, 4). \end{aligned}$$

$\theta_i(w; *)$ is a parallel section defined on the open right half-plane of the expression parameters, but the expression parameter τ turns out to give quasi-periodicity with the exponential factor $e^{2iw-\tau}$.

Noting that $(e_*^{2iw}-1) * \theta_3(w, *) = 0$ in the computation of $*_\tau$ -product, we have

Proposition 1.3 *If $f \in \operatorname{Hol}(\mathbb{C})$ satisfies $f(w+\pi) = f(w)$ and $:(e_*^{2iw}-1):_\tau *_\tau f = 0$, then*

$$f = c : \theta_3(w, *) :_\tau, \quad c \in \mathbb{C}.$$

Proof By periodicity, the Fourier expansion theorem gives $f(w) = \sum a_n e^{2inw}$, but by the formula of $*$ -exponential functions, this is rewritten as $f(w) = : \sum c_n e_*^{2inw} :_\tau$. This gives the result, for the second identity gives that $c_{n+1} = c_n$. \square

1.4.1 Two different inverses of an element

The trivial identities of theta functions give many inverses. The convergence of bilateral geometric series gives a little strange feature. Note that if $\operatorname{Re} \tau > 0$, then τ -expressions of $\sum_{n=0}^{\infty} e_*^{2niw}$ and $-\sum_{n=-\infty}^{-1} e_*^{2niw}$ both converge in $\operatorname{Hol}(\mathbb{C})$ to give inverses of the element $:(1-e_*^{2iw}):_\tau$, and $\theta_3(w, \tau)$ is the difference of these inverses.

We denote these inverses by using short notations:

$$(1-e_*^{2iw})_{*+}^{-1} = \sum_{n=0}^{\infty} e_*^{2niw}, \quad (1-e_*^{2iw})_{*-}^{-1} = -\sum_{n=1}^{\infty} e_*^{-2niw}, \quad (1-e_*^{-2iw})_{*+}^{-1} = \sum_{n=1}^{\infty} e_*^{-2niw}$$

Apparently, this breaks associativity:

$$\left((1-e_*^{2iw})_{*+}^{-1} *_\tau (1-e_*^{2iw}) \right) *_\tau (1-e_*^{2iw})_{*-}^{-1} \neq (1-e_*^{2iw})_{*+}^{-1} *_\tau \left((1-e_*^{2iw}) *_\tau (1-e_*^{2iw})_{*-}^{-1} \right).$$

Similarly, $\theta_4(w, *)$ is the difference of two inverses of $1+e_*^{2iw}$

$$(1+e_*^{2iw})_{*+}^{-1} = \sum_{n=0}^{\infty} (-1)^n e_*^{2niw}, \quad (1+e_*^{2iw})_{*-}^{-1} = - \sum_{n=1}^{\infty} (-1)^n e_*^{-2niw}.$$

Note that $2e_*^{iw} * \sum_{n \geq 0} (-1)^n e_*^{2inw}$ and $2e_*^{-iw} * \sum_{n \geq 0} (-1)^n e_*^{-2inw}$ are $*$ -inverses of $\frac{1}{2}(e_*^{iw} + e_*^{-iw})$. We denote these by

$$(\cos_* w)_{*+}^{-1}, \quad (\cos_* w)_{*-}^{-1}.$$

Then we see

$$2i\theta_1(w, *) = (\cos_* w)_{*+}^{-1} - (\cos_* w)_{*-}^{-1}.$$

Similarly, $2ie_*^{-iw} * \sum_{n \geq 0} e_*^{-2inw}$, $-2ie_*^{iw} * \sum_{n \geq 0} e_*^{2inw}$ are both $*$ -inverses of $\frac{1}{2i}(e_*^{iw} - e_*^{-iw})$. We denote these by

$$(\sin_* w)_{*+}^{-1}, \quad (\sin_* w)_{*-}^{-1}.$$

Then we see

$$2i\theta_2(w, *) = (\sin_* w)_{*+}^{-1} - (\sin_* w)_{*-}^{-1}.$$

Every $\theta_i(w, *)$ is written as the difference of two different inverses of suitable linear combinations of $*$ -exponential functions.

As the associativity fails, $\theta_i(w, *)$ may be viewed as associators. But, these are not elements of non-associative *algebra*, for the product $(\sum_{n=0}^{\infty} e_*^{niw}) * (\sum_{n=-\infty}^{-1} e_*^{niw})$ is not defined as it diverges apparently.

1.4.2 Star-delta functions

Next, we note a similar phenomenon as above for the generator of the algebra:

Proposition 1.4 *If $\operatorname{Re} \tau > 0$, then for every $a \in \mathbb{C}$, the integrals*

$$i \int_{-\infty}^0 :e_*^{it(a+w)}:_{\tau} dt, \quad -i \int_0^{\infty} :e_*^{it(a+w)}:_{\tau} dt$$

converge in $\operatorname{Hol}(\mathbb{C})$ to give inverses of $a+w$. If $a = -i\alpha$, then

$$\int_{-\infty}^0 :e_*^{t(\alpha+iw)}:_{\tau} dt, \quad - \int_0^{\infty} :e_*^{t(\alpha+iw)}:_{\tau} dt$$

converge in $\operatorname{Hol}(\mathbb{C})$ to give inverses of $\alpha+iw$.

Proof For the same reason as in the case of θ_3 , we get the convergence. To confirm these give inverses of $a+w$, we compute by the continuity of $i(a+w)*$ as follows:

$$(a+w)*i \int_{-\infty}^0 e_*^{it(a+w)} dt = \int_{-\infty}^0 i(a+w)*e_*^{it(a+w)} dt = \int_{-\infty}^0 \frac{d}{dt} e_*^{it(a+w)} dt = 1.$$

The others are obtained similarly. □

Denote these inverses by

$$\begin{aligned} : (a+w)_{*+}^{-1} :_{\tau} &= i \int_{-\infty}^0 : e_*^{it(a+w)} :_{\tau} dt, & : (a+w)_{*-}^{-1} :_{\tau} &= -i \int_0^{\infty} : e_*^{it(a+w)} :_{\tau} dt, & \operatorname{Re} \tau > 0. \\ : (\alpha+iw)_{*+}^{-1} :_{\tau} &= \int_{-\infty}^0 : e_*^{t(\alpha+iw)} :_{\tau} dt, & : (\alpha+iw)_{*-}^{-1} :_{\tau} &= - \int_0^{\infty} : e_*^{t(\alpha+iw)} :_{\tau} dt, & \operatorname{Re} \tau > 0. \end{aligned}$$

The definitions of $:(-i\alpha+w)_{*+}^{-1}:$ and $:(\alpha+iw)_{*+}^{-1}:$ are the same, since these have the same τ -expressions. As a result, we have the identity

$$(-i\alpha+w)_{*+}^{-1} = (-i)^{-1}(\alpha+iw)_{*+}^{-1} = i(\alpha+iw)_{*+}^{-1}.$$

But this is not a special case of $\frac{e^{i\theta}}{e^{i\theta}(a+w)} = \frac{1}{a+w}$, which is trivial in ordinary calculus. Such an identity cannot be applied in the case where $e^{i\theta}$ causes a rotation of the path of integration. (See § 1.4.4.)

The difference of these two inverses is

$$(1.19) \quad (a+w)_{*+}^{-1} - (a+w)_{*-}^{-1} = i \int_{-\infty}^{\infty} e_*^{it(a+w)} dt, \quad \operatorname{Re} \tau > 0.$$

The right hand side may be viewed as a delta function in the world of $*$ -functions.

We define the $*$ -delta function by

$$(1.20) \quad \delta_*(a+w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e_*^{it(a+w)} dt \quad \operatorname{Re} \tau > 0.$$

It is easy to see that $(a+w)*\delta_*(a+w) = 0$. In contrast, an element e_*^{2iw} cannot be a zero-divisor:

Proposition 1.5 $:e_*^{2iw} :_{\tau} *_{\tau} f(w) = e^{-\tau} e^{2iw} f(w+i\tau)$. Hence a real analytic function $f(w)$ satisfying $:e_*^{2iw} :_{\tau} *_{\tau} f(w) = 0$ must be 0.

In ordinary calculus, $\int_{-\infty}^{\infty} e^{it(a+x)} dt = 2\pi\delta(a+x)$ is not a function but a distribution. On the other hand, in the world of $*$ -functions, the τ -expression $:\delta_*(a+w):_{\tau}$ of $\delta_*(a+w)$ is an entire function:

$$(1.21) \quad :\delta_*(a+w):_{\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{4}t^2\tau} e^{it(a+w)} dt = \frac{1}{\sqrt{\pi\tau}} e^{-\frac{1}{\tau}(a+w)^2}, \quad \operatorname{Re} \tau > 0, \quad a \in \mathbb{C}$$

This is obtained directly as follows: Set

$$-\frac{1}{4}t^2\tau + it(a+w) = -\frac{1}{4}\tau(t - \frac{2}{\tau}(a+w))^2 - \frac{1}{\tau}(a+w)^2.$$

Then we have

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}t^2\tau} e^{it(a+w)} dt = \int_{-\infty}^{\infty} e^{-\frac{\tau}{4}(t-\frac{2i}{\tau}(a+w))^2} dt e^{-\frac{1}{\tau}(a+w)^2}.$$

Note now that the integral $\int_{-\infty}^{\infty} e^{-\frac{\tau}{4}(t-\alpha)^2} dt$ converges for every $\alpha \in \mathbb{C}$, and integration by parts gives

$$\frac{d}{d\alpha} \int_{-\infty}^{\infty} e^{-\frac{\tau}{4}(t-\alpha)^2} dt = - \int_{-\infty}^{\infty} \frac{d}{dt} e^{-\frac{\tau}{4}(t-\alpha)^2} dt = 0.$$

Thus, $\int_{-\infty}^{\infty} e^{-\frac{\tau}{4}(t-\frac{2}{\tau}(a+w))^2} dt$ is obtained by the value at $a+w=0$, as $\int_{-\infty}^{\infty} e^{-\frac{\tau}{4}t^2} dt = 2 \int_0^{\infty} e^{-\frac{\tau}{4}t^2} dt$.

The world of $*$ -functions is a deformed version of ordinary world, and as a result one can see what is not able to see in the ordinary world. Note that there are many inverses, for $(a+w)_{*+}^{-1} + Ci\delta_*(a+w)$ are all inverses of $a+w$ for any C .

1.4.3 Jacobi's imaginary transformations

By formula (1.21), we see the following series

$$\begin{aligned} \tilde{\theta}_1(w, *) &= \sum_n (-1)^n \delta_*(w + \frac{\pi}{2} + \pi n), & \tilde{\theta}_2(w, *) &= \sum_n (-1)^n \delta_*(w + \pi n) \\ \tilde{\theta}_3(w, *) &= \sum_n \delta_*(w + \pi n), & \tilde{\theta}_4(w, *) &= \sum_n \delta_*(w + \frac{\pi}{2} + \pi n), \end{aligned}$$

converge in the τ -expression for $\text{Re } \tau > 0$. These may be viewed as π -periodic/ π -alternating periodic $*$ -delta functions on \mathbb{R} .

As $e^{2\pi i n}=1$, we have identities

$$e_*^{2iw} * \tilde{\theta}_i(w, *) = \tilde{\theta}_i(w, *), \quad (i = 2, 3), \quad e_*^{2iw} * \tilde{\theta}_i(w, *) = -\tilde{\theta}_i(w, *), \quad (i = 1, 4).$$

By a slight modification of Proposition 1.3, we have

$$\theta_i(w, *) = \alpha_i \tilde{\theta}_i(w, *), \quad \alpha_i \in \mathbb{C}.$$

Note that α_i does not depend on the expression parameter τ . Taking the τ -expressions of both sides at $\tau=\pi$ and setting $w=0$, we have $\alpha_i = 1/2$.

Proposition 1.6 $\theta_i(w, *) = \frac{1}{2} \tilde{\theta}_i(w, *)$ for $i = 1 \sim 4$.

The Jacobi's imaginary transformation is given by taking the τ -expression of these identities.

This may be proved directly by the following manner: Since $f(t) = \sum_n e_*^{2(n+t)iw}$ is periodic function of period 1, Fourier expansion formula gives

$$f(t) = \sum_m \int_0^1 f(s) e^{-2\pi i m s} ds e^{2\pi i m t}, \quad \theta_3(w, *) = f(0) = \sum_m \int_0^1 \left(\sum_n e_*^{2(n+s)iw} \right) e^{-2\pi i m s} ds.$$

Since $e^{-2\pi i m s} = e^{-2\pi i m(s+n)}$, we have

$$\begin{aligned} f(0) &= \sum_m \int_0^1 \left(\sum_n e_*^{2(n+s)iw} e^{-2(n+s)i\pi m} \right) ds \\ &= \sum_m \int_{-\infty}^{\infty} e_*^{2si(w+\pi m)} ds = \frac{1}{2} \sum_m \delta_*(w + \pi m). \end{aligned}$$

For instance, we have

$$\begin{aligned} (1.22) \quad \theta_3(w, \tau) &= \frac{2\pi}{2} : \sum_n \delta_*(w + \pi n) :_{\tau} = \sqrt{\frac{\pi}{\tau}} \sum_n e^{-\frac{1}{\tau}(w + \pi n)^2} \\ &= \sqrt{\frac{\pi}{\tau}} e^{-\frac{1}{\tau}w^2} \sum_n e^{-\pi^2 n^2 \tau^{-1} - 2\pi n \tau^{-1} w} = \sqrt{\frac{\pi}{\tau}} e^{-\frac{1}{\tau}w^2} \theta_3\left(\frac{\pi w}{i\tau}, \frac{\pi^2}{\tau}\right) \end{aligned}$$

This is remarkable since a relation of two different expressions (viewpoints) are explicitly given.

In particular, Jacobi's theta relation is obtained by setting $w = 0$ in (1.22):

$$(1.23) \quad \theta_3(0, \tau) = \sqrt{\frac{\pi}{\tau}} \theta_3\left(0, \frac{\pi^2}{\tau}\right).$$

This will be used to obtain the functional identities of the $*$ -zeta function in the forthcoming paper.

1.4.4 The boundary of expression parameters

First of all, recall that Fabry-Pólya's gap theorem gives:

Theorem 1.2 *For a series $\sum_{n=0}^{\infty} a_n q^{n^2}$ suppose $a_n \geq 0$ and the radius of convergence is 1, e.g. the case $\limsup_n |a_n| = 1$. Then the circle $|q|=1$ is the natural boundary.*

Fabry-Pólya's gap Theorem 1.2 shows that the expression parameters of $*$ -functions such as Jacobi's theta functions are strictly limited in the right half plane, that is, complex rotations of the domain is forbidden.

1.4.5 Integration vs discrete summation

For a while, the expression parameter is restricted in the domain $\operatorname{Re} \tau < 0$. The difference between the integral $\int_{-\infty}^0 e_*^{tw} dt$ and the discrete sum $\sum_{n=0}^{\infty} e_*^{nw}$ appears in the following logarithmic derivative $(\frac{\partial}{\partial \beta} \log) f(\beta) = f'(\beta) * f(\beta)^{-1}$:

$$(1.24) \quad \sum_{k=1}^{\infty} \left(\frac{\partial}{\partial \beta} \log \right) \int_{-\infty}^0 e_*^{t\beta k w} dt, \quad \sum_{k=1}^{\infty} \left(\frac{\partial}{\partial \beta} \log \right) \sum_{n=1}^{\infty} e_*^{n\beta k w}.$$

Since integration by parts gives

$$\frac{\partial}{\partial \beta} \int_{-\infty}^0 e_*^{t\beta kw} dt = \beta^{-1} \int_{-\infty}^0 t(k\beta w) * e_*^{t\beta kw} dt = -\beta^{-1} \int_{-\infty}^0 e_*^{t\beta kw} dt,$$

the first term of (1.24) is $\sum_{k=1}^{\infty} \beta^{-1}$ and there is no way to avoid divergence. However, the second term of (1.24) is rewritten as $\sum_{k=1}^{\infty} (\sum_{n=1}^{\infty} e_*^{n\beta k}) * kw$ by using the identity

$$\frac{\partial}{\partial \beta} \sum_{n=1}^{\infty} e_*^{n\beta kw} = kw * \sum_{n=1}^{\infty} n e_*^{n\beta kw} = kw * \left(\sum_{n=1}^{\infty} e_*^{n\beta kw} \right) * \left(\sum_{n=1}^{\infty} e_*^{n\beta kw} \right)$$

which is easily proved. The τ -expression of $\sum_{k=1}^{\infty} (\sum_{n=1}^{\infty} e_*^{n\beta k}) * kw$ converges absolutely when $\operatorname{Re} \tau < 0$ by the next Lemma:

Lemma 1.2 *If $\operatorname{Re} \tau < 0$, then the τ -expression of $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} k e_*^{n\beta kw}$ converges absolutely.*

Proof The τ -expression is given by $:e_*^{n\beta kw}:_{\tau} = e^{nk\beta w + \frac{1}{4}(nk\beta)^2 \tau}$. Rewriting this as

$$nk\beta w + \frac{1}{4}(nk\beta)^2 \tau = \frac{\tau}{4} \left(\beta nk + \frac{2}{\tau} w \right)^2 - \frac{1}{\tau} w^2$$

and assuming $\operatorname{Re}(\tau\beta^2) < 0$, we only have to show the convergence of $\sum_{k,n=1}^{\infty} k e^{\frac{\tau}{4}(\beta nk + \frac{2}{\tau} w)^2}$. Restricting w to any compact subset, it is enough to show the absolute convergence of $\sum_{k,n=1}^{\infty} k e^{\frac{\tau}{4}\beta^2(nk+K)^2}$ for every $K > 0$.

For $n, k \geq 1$, $(nk+K)^2 \leq (n^2+1)^2 + (k^2+1)^2 + K^2$ gives

$$\sum_{k,n=1}^{\infty} k e^{\frac{\tau}{4}\beta^2(nk+K)^2} \leq e^{\frac{1}{4}\tau\beta^2 K^2} \sum_{n=1}^{\infty} e^{\frac{1}{4}\tau\beta^2(n^2+1)^2} \sum_{k=1}^{\infty} k e^{\frac{1}{4}\tau\beta^2(k^2+1)^2}.$$

This gives convergence. □

2 Calculus of inverses

In this section, we give several ways of treatment of $*$ -product of inverse elements. First of all, we note that the elementary method of constant variation gives many inverses of a single element. Let $a \in \mathbb{C}$. By the product formula, $(a+w)*$ is viewed as a linear operator of $\operatorname{Hol}(\mathbb{C})$ into itself. Suppose $\tau \neq 0$. Then, $(a+w) *_{\tau} f(w) = 0$ is a differential equation

$$(a+w)f(w) + \frac{\tau}{2} \partial_w f(w) = 0.$$

Solving this, we have

$$(2.1) \quad (a+w) *_{\tau} C e^{-\frac{1}{\tau}(a+w)^2} = C e^{-\frac{1}{\tau}(a+w)^2} *_{\tau} (a+w) = 0.$$

The method of constant variation gives a function $g_a(w)$ such that

$$(a+w)*_{\tau}g_a(w) = g_a(w)*_{\tau}(a+w) = 1.$$

Thus, we have

$$(2.2) \quad g_a(w) = \frac{2}{\tau} \int_0^1 e^{\frac{1}{\tau}((a+wt)^2 - (a+w)^2)} w dt + C e^{-\frac{1}{\tau}(a+w)^2}.$$

This breaks associativity

$$(e^{-\frac{1}{\tau}(a+w)^2} *_{\tau}(a+w)) *_{\tau} g_a(w) \neq e^{-\frac{1}{\tau}(a+w)^2} *_{\tau} ((a+w) *_{\tau} g_a(w)).$$

The method of constant variation is a strange method from a viewpoint of operators.

Note that if $a \neq b$, then 4 elements with independent \pm -sign

$$\frac{1}{b-a}(((a+w)_{*\pm}^{-1} - (b+w)_{*\pm}^{-1}))$$

give respectively $*$ -inverses of $(a+w)*(b+w)$. Thus, we define $*$ -inverse with independent \pm -sign by

$$(2.3) \quad (a+w)_{*\pm}^{-1} * (b+w)_{*\pm}^{-1} = \frac{1}{b-a}(((a+w)_{*\pm}^{-1} - (b+w)_{*\pm}^{-1})).$$

By providing $a \neq b$, a direct calculation of $*$ -product by (2.3) gives

$$((a+w)_{*+}^{-1} - (a+w)_{*-}^{-1}) * ((b+w)_{*+}^{-1} - (b+w)_{*-}^{-1}) = 0, \quad a, b \in \mathbb{C}, \quad a \neq b.$$

Theorem 2.1 Suppose $a+w$ has two different $*$ -inverses $(a+w)_{*+}^{-1}$ and $(a+w)_{*-}^{-1}$, then

$$(a+w) * ((a+w)_{*+}^{-1} - (a+w)_{*-}^{-1}) = 0.$$

If $b+w$ has also two different $*$ -inverses, then by providing $a \neq b$, the $*$ -product

$$((a+w)_{*+}^{-1} - (a+w)_{*-}^{-1}) * ((b+w)_{*+}^{-1} - (b+w)_{*-}^{-1}) = 0, \quad a \neq b.$$

In the later section, we show the formula

$$\delta_*(a+w) * \delta_*(b+w) = \delta(a-b) \delta_*(b+w).$$

Existence of two different inverses means that there are two ways to define *logarithmic derivatives*:

$$\frac{d}{d\alpha} \log_{*+}(\alpha+iw) = (\alpha+iw)_{*+}^{-1}, \quad \frac{d}{d\alpha} \log_{*-}(\alpha+iw) = (\alpha+iw)_{*-}^{-1},$$

while $\frac{d}{d\alpha} \log_{*\pm} e_*^{(\alpha+iw)} = \alpha+iw$.

Note also that if $\operatorname{Re} \tau > 0$, then $(z+w)_{*\pm}^{-1}$ is an entire function with respect to z , and hence the integral on any closed path C

$$\int_C f(z) (z+w)_{*\pm}^{-1} dz = 0$$

for every entire function $f(z)$. Hence, we cannot use Cauchy's integral formula.

2.1 Resolvent calculus

Products of inverses are given for $k, \ell \geq 0$ by

$$(2.4) \quad (\alpha+w)_{*+}^{-k-1} * (\beta+w)_{*+}^{-\ell-1} = (-1)^{k+\ell} \frac{1}{k!\ell!} \frac{d^{k+\ell}}{d\alpha^k d\beta^\ell} \frac{1}{\beta-\alpha} ((\alpha+w)_{*+}^{-1} - (\beta+w)_{*+}^{-1}).$$

Note that $(\alpha+w)_{*+}^{-1} * (\alpha+w)_{*+}^{-1}$ is defined by this formula. Such calculations of $*$ -inverses will be referred as the **resolvent calculus**.

We now make $*$ -inverses of $p_*(w)$ by the partial fraction decomposition as follows: Setting $X = w$ for simplicity and regarding $p(X)$ is an ordinary polynomial, we see

$$\frac{1}{p(X)} = \sum_{k=1}^R \frac{c_k}{(a_k - X)^{m_k}}, \quad c_k \in \mathbb{C}.$$

Since $\mathbb{C}[w_*]$ is isomorphic to the ordinary polynomial ring $\mathbb{C}[X]$, we see

$$(2.5) \quad p_*(w)_{*\pm}^{-1} = \sum_{k=1}^R c_k (a_k - w)_{*\pm}^{-m_k}$$

is well-defined by (2.4) to give $*$ -inverses of $p_*(w)$. That is, every $*$ -polynomial has $*$ -inverses.

In these computations, the following proposition is very useful:

Proposition 2.1 *Suppose w is restricted to a compact domain. If $\operatorname{Re} \tau > 0$, then $:(z+w)_{*\pm}^{-1}:\tau$ are both rapidly decreasing functions w.r.t. z on the closed lower/upper complex half-plane.*

Proof Set $z = x + iy$. We will show that $z^k \partial_z^\ell : (z+w)_{*+}^{-1} : \tau$ are bounded on the closed lower half-plane.

$$z^k \partial_z^\ell : (z+w)_{*+}^{-1} : \tau = i \int_{-\infty}^0 (it)^\ell ((-i\partial_t)^k e^{itz}) e^{-\frac{1}{4}\tau t^2} e^{itw} dt.$$

Integration by parts gives that there is a polynomial $p(t, w)$ depending on k, ℓ such that

$$z^k \partial_z^\ell : (z+w)_{*+}^{-1} : \tau = i \int_{-\infty}^0 p(t, w) e^{itz} : e_*^{itw} : \tau dt.$$

If z is in the lower half-plane then $|e^{itz}| \leq 1$. The boundedness follows. The other case is proved similarly. \square .

Hence we have the same result as in (1.21):

Corollary 2.1 *Suppose $\operatorname{Re} \tau > 0$. Then $\delta_*(z+w)$ is rapidly decreasing on the real line.*

Since $(-z+w)_{*-}^{-1} = -(z-w)_{*-}^{-1}$ is rapidly decreasing w.r.t. z on the right half-plane, we often write these as follows:

$$(z-w)_{*-}^{-1} = i \int_{-\infty}^0 e_*^{it(z-w)} dt,$$

or similarly

$$(z+w)_{*+}^{-1} = -i \int_0^\infty e_*^{-it(z+w)} dt.$$

Note that although $(\alpha+w)_{*+}^{-1} * (\alpha+w)_{*-}^{-1}$ diverges, the product $(\alpha+w)_{*+}^{-k} * (\beta+w)_{*-}^{-\ell}$ is defined, if $\alpha \neq \beta$, as

$$(-1)^{k+\ell} \frac{1}{k!\ell!} \frac{d^{k+\ell}}{d\alpha^k d\beta^\ell} \frac{1}{\beta-\alpha} ((\alpha+w)_{*+}^{-1} - (\beta+w)_{*-}^{-1}).$$

To compute the $*$ -product of mixed elements, we first replace $(\alpha+w)_{*-}^{-1}$ by $(\alpha+w)_{*+}^{-1} - \delta_*(\alpha+w)$, and then we have only to note the following a little strange identities:

$$(2.6) \quad \delta_*(\alpha+w) * \delta_*(\beta+w) = 0, \quad \alpha \neq \beta, \quad \text{cf. (3.5)}$$

$$(2.7) \quad (\alpha+w)_{*+}^{-1} * \delta_*(\beta+w) = \frac{1}{\alpha-\beta} \delta_*(\beta+w) \quad (\alpha \neq \beta).$$

Since $\delta_*(a+w) * w = -a\delta_*(a+w) = w * \delta_*(a+w)$, (2.6) barely protect the associativity

$$(\delta_*(a+w) * w) * \delta_*(b+w) = \delta_*(a+w) * (w * \delta_*(b+w)).$$

We prove these formulas directly by integration. In what follows, we assume $\text{Re}(a-b) > 0$, but this is not essential:

$$\begin{aligned} (a+w)_{*+}^{-1} * (b+w)_{*+}^{-1} &= - \int_{-\infty}^0 \int_{-s'}^{s'} e^{\frac{1}{2}i\sigma(a-b)} d\sigma e^{\frac{1}{2}is'(a+b)} e^{is'w} ds' \\ &= \frac{1}{b-a} ((a+w)_{*+}^{-1} - (b+w)_{*+}^{-1}). \\ (a+w)_{*+}^{-1} * (b+w)_{*-}^{-1} &= (a+w)_{*+}^{-1} * ((b+w)_{*+}^{-1} - \delta_*(b+w)) \\ (a+w)_{*+}^{-1} * \delta_*(b+w) &= \int_{-\infty}^0 \int_{\mathbb{R}} e^{i(a-b)t'} e^{is'b} e_*^{is'w} dt' ds' = \frac{1}{a-b} \delta_*(b+w). \end{aligned}$$

2.2 The case of inverses by discrete summations

Suppose $\text{Re } \tau > 0$, then for every $a \in \mathbb{C}$, the bilateral geometric series $\sum_{n \in \mathbb{Z}} :e_*^{2in(w+a)} :_\tau$ converges. Denote this by $\theta_3(w+a, *)$. We see easily that

$$\theta_3(w+a, *) = (1 - e^{2ia} e_*^{2iw})_{*+}^{-1} - (1 - e^{2ia} e_*^{2iw})_{*-}^{-1}.$$

For another $b \in \mathbb{C}$, we make $\theta_3(w+b, *)$ and

$$\theta_3(w+b, *) = (1 - e^{2ib} e_*^{2iw})_{*+}^{-1} - (1 - e^{2ib} e_*^{2iw})_{*-}^{-1}.$$

As e^{2ia} , e^{2ib} are scalars, we see that 4 independent elements

$$\frac{1}{e^{2ia}-e^{2ib}} \left((1-e^{2ia}e_*^{2iw})_{*\pm}^{-1} - (1-e^{2ib}e_*^{2iw})_{*\pm}^{-1} \right) \quad (\text{independent } \pm \text{ sign})$$

gives respectively inverses of $(1-e^{2ia}e_*^{2iw})_{*\pm}(1-e^{2ib}e_*^{2iw})_{*\pm}^{-1}$. Hence, we define the $*$ -product of inverses by such resolvent calculations, that is, we define

$$(1-e^{2ia}e_*^{2iw})_{*\pm}^{-1} * (1-e^{2ib}e_*^{2iw})_{*\pm}^{-1} = \frac{1}{e^{2ia}-e^{2ib}} \left((1-e^{2ia}e_*^{2iw})_{*\pm}^{-1} - (1-e^{2ib}e_*^{2iw})_{*\pm}^{-1} \right),$$

although there are many choice of inverses.

Theorem 2.2 *If $a \neq b$, then under such a definition of $*$ -product of inverses, we have $\theta_3(w+a, *) * \theta_3(w+b, *) = 0$. In particular $\theta_3(w, *) * \theta_4(w, *) = 0$ by noting $\theta_3(w+\frac{\pi}{2}, *) = \theta_4(w, *)$. It follows by differentiating*

$$(\partial_w \theta_3(w, *)) * \theta_4(w, *) + \theta_3(w, *) * \partial_w \theta_4(w, *) = 0.$$

Proof Using $\theta_3(w+a, *) = (1-e^{2ia}e_*^{2iw})_{*+}^{-1} - (1-e^{2ia}e_*^{2iw})_{*-}^{-1}$, we compute the $*$ -product

$$\left((1-e^{2ia}e_*^{2iw})_{*+}^{-1} - (1-e^{2ia}e_*^{2iw})_{*-}^{-1} \right) * \left((1-e^{2ib}e_*^{2iw})_{*+}^{-1} - (1-e^{2ib}e_*^{2iw})_{*-}^{-1} \right)$$

by the resolvent calculus. By multiplying $e^{2ia}-e^{2ib}$, we see

$$\begin{aligned} & (1-e^{2ia}e_*^{2iw})_{*+}^{-1} - (1-e^{2ib}e_*^{2iw})_{*+}^{-1} \\ & - (1-e^{2ia}e_*^{2iw})_{*-}^{-1} + (1-e^{2ib}e_*^{2iw})_{*-}^{-1} \\ & - (1-e^{2ia}e_*^{2iw})_{*+}^{-1} + (1-e^{2ib}e_*^{2iw})_{*+}^{-1} \\ & + (1-e^{2ia}e_*^{2iw})_{*-}^{-1} - (1-e^{2ib}e_*^{2iw})_{*-}^{-1}. \end{aligned}$$

We see all terms are cancelled out. □

The trivial identity gives $e_*^{2iw} * \theta_3(w+a, *) = \theta_3(w+a, *) * e_*^{2iw} = e^{-2ia} \theta_3(w+a, *)$ and the vanishing Theorem 2.2 barely protects the associativity:

$$\left(\theta_3(w+a, *) * e_*^{2iw} \right) * \theta_3(w+b, *) = \theta_3(w+a, *) * \left(e_*^{2iw} * \theta_3(w+b, *) \right), \quad a \neq b.$$

We keep the expression parameter τ so that $\text{Re } \tau > 0$. Let $\mathbb{C}[e_*^{iw}]$ be the polynomial ring of e_*^{iw} . For every monic polynomial $p(X)$ (with 1 as the coefficient of the highest degree), we set by unique factorization theorem

$$p(e_*^{iw}) = \prod_{k=1}^N (a_k - e_*^{iw})^{\ell_k}.$$

For a special polynomial X^n , we use the notation $I^{(n)}(e_*^{iw}) = e_*^{niw}$. Note that

$$(2.8) \quad (1-X)_{*\pm}^{-n-1} = \frac{1}{(n-1)!} \partial_X^n (1-X)_{*\pm}^{-1}$$

are well-defined to give inverses of $(1-X)_*^{n+1}$.

We now make $*$ -inverses of $p(e_*^{iw})$ by the partial fraction decomposition as follows: Setting $X = e_*^{iw}$ for simplicity and regarding $p(X)$ is an ordinary polynomial, we see

$$\frac{1}{p(X)} = \sum_{k=1}^R \frac{c_k}{(a_k - X)^{m_k}}, \quad c_k \in \mathbb{C}.$$

Since $\mathbb{C}[e_*^{iw}]$ is isomorphic to the ordinary polynomial ring $\mathbb{C}[X]$, we see

$$p(e_*^{iw})_{*\pm}^{-1} = \sum_{k=1}^R c_k (a_k - e_*^{iw})_{*\pm}^{-m_k}$$

is well-defined by (2.8) to give $*$ -inverses of $p(e_*^{iw})$, where we have to note that

$$(e_*^{iw})_{*+}^{-1} = e_*^{-iw} = (e_*^{iw})_{*-}^{-1}, \quad (\text{cf. Proposition 1.5})$$

while in the continuous case, Proposition 1.4 shows that w has two different inverses w_{*+}^{-1} , w_{*-}^{-1} .

As in the case of $*$ -theta functions we define

$$\hat{P}_*(e_*^{iw}) = p(e_*^{iw})_{*+}^{-1} - p(e_*^{iw})_{*-}^{-1}, \quad \hat{I}_*^{(n)}(e_*^{iw}) = 0.$$

This is 2π -periodic, and providing $p(0) = 1$ and setting $p(e_*^{iw}) = 1 - q_*(e_*^{iw})$, we see

$$q_*(e_*^{iw}) * \hat{P}_*(e_*^{iw}) = \hat{P}_*(e_*^{iw}).$$

As a Corollary of Fabry-Pólya's gap Theorem 1.2 we see the following:

Corollary 2.2 *For every polynomial $p(X)$, $:p(e_{*\pm}^{iw})^{-1}:_\tau$ are holomorphic w.r.t. τ on the right half-plane $\text{Re } \tau > 0$, and $\text{Re } \tau = 0$ is the natural boundary except the case $p(X) = aX^n$.*

If another polynomial $q(X)$ is relatively prime to $p(X)$, then there are polynomials $a(X)$, $b(X)$ such that $a(X)p(X) + b(X)q(X) = 1$. Hence, we have that 4-independent elements

$$b(e_*^{iw}) * p(e_*^{iw})_{*\pm}^{-1} + a(e_*^{iw}) * q(e_*^{iw})_{*\pm}^{-1}$$

are inverses of $p(e_*^{iw}) * q(e_*^{iw})$ respectively. Hence, we define the $*$ -products of these inverses by

$$p(e_*^{iw})_{*\pm}^{-1} * q(e_*^{iw})_{*\pm}^{-1} = b(e_*^{iw}) p(e_*^{iw})_{\pm}^{-1} + a(e_*^{iw}) q(e_*^{iw})_{\pm}^{-1}$$

by using resolvent calculations. We have then a little strange theorem

Theorem 2.3 *If $p(X)$, $q(X)$ are relatively prime, that is, the resultant of $p(X)$ and $q(X)$ does not vanish, then the $*$ -product $\hat{P}_*(e_*^{iw}) * \hat{Q}_*(e_*^{iw}) = 0$.*

Proof is the same as in Theorem 2.2. We compute as follows:

$$\begin{aligned} \left(p(e_*^{iw})_{*+}^{-1} - p(e_*^{iw})_{*-}^{-1} \right) * \left(q(e_*^{iw})_{*+}^{-1} - q(e_*^{iw})_{*-}^{-1} \right) = & b(e_*^{iw}) p(e_*^{iw})_{*+}^{-1} + a(e_*^{iw}) q(e_*^{iw})_{*+}^{-1} \\ & - b(e_*^{iw}) p(e_*^{iw})_{*-}^{-1} - a(e_*^{iw}) q(e_*^{iw})_{*-}^{-1} \\ & - b(e_*^{iw}) p(e_*^{iw})_{*-}^{-1} - a(e_*^{iw}) q(e_*^{iw})_{*+}^{-1} \\ & + b(e_*^{iw}) p(e_*^{iw})_{*-}^{-1} + a(e_*^{iw}) q(e_*^{iw})_{*-}^{-1} = 0. \end{aligned}$$

□

This sounds strange because the condition for the resultant is an open condition, while the conclusion looks like a closed condition.

Note In conformal field theory, a formal distribution $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ is often used. As we have seen, this is written as the difference of two inverses $\delta(z) = (1-z)_+^{-1} - (1-z)_-^{-1}$ where

$$(1-z)_+^{-1} = \frac{1}{1-z}, \quad (1-z)_-^{-1} = -\frac{z^{-1}}{1-z^{-1}}.$$

We may extend this to define $\delta(az) = \sum_{n \in \mathbb{Z}} (az)^n$ for $a \in \mathbb{C}$. Then, we easily see that

$$\delta(az) = (1-az)_+^{-1} - (1-az)_-^{-1}$$

and if $a \neq b$, then $\delta(az)\delta(bz) = 0$ by the same calculation as in Theorem 2.2 below.

Since the trivial identity $az\delta(az) = \delta(az)$ gives $z\delta(az) = a^{-1}\delta(az)$, the associativity holds

$$a^{-1}\delta(az)\delta(bz) = (\delta(az)z)\delta(bz) = \delta(az)(z\delta(bz)) = b^{-1}\delta(az)\delta(bz).$$

2.2.1 Half-series algebra

It is well known that if a formal power series satisfies $\sum_{n=0}^{\infty} a_n z^n = 0$, then $a_n = 0$. This is proved by setting $z=0$ to get $a_0 = 0$, and then taking ∂_z followed by evaluating at $z = 0$ gives $a_1 = 0$ and so on. Hence this method cannot be applied to formal power series $\sum_{n=0}^{\infty} a_n e_*^{niw}$.

We assume $\text{Re } \tau > 0$ throughout this subsection. A formal power series

$$z^\ell \sum_{n=0}^{\infty} a_n z^n, \quad \ell \in \mathbb{Z}$$

is called a *convergent power series*, if $\sum_{n=0}^{\infty} a_n z^n$ has a positive radius of convergence. It is easy to show that if $z^\ell \sum_{n=0}^{\infty} a_n z^n$ is a convergent power series, then

$$f(w) = :e_*^{\ell iw} * \sum_{n=0}^{\infty} a_n e_*^{niw} :_\tau$$

is an entire function of w . Hence if $f(w) = 0$, then Proposition 1.1 gives $\sum_{n=0}^{\infty} a_n e_*^{niw} :_{\tau} = 0$, and $a_0 = 0$ by taking $w \rightarrow i\infty$. Thus, the repeated use of Proposition 1.1 gives all $a_n = 0$.

Note that the product of two convergent power series is a convergent power series. If $z^{\ell} \sum_{n=0}^{\infty} a_n z^n$, ($\ell \in \mathbb{Z}$) is a convergent power series, then its inverse $z^{-\ell} (\sum_{n=0}^{\infty} a_n z^n)^{-1}$ obtained by the method of indeterminate constants is also a convergent power series.

We denote by \mathfrak{H}_+ be the space of power series $:e_*^{\ell iw} * \sum_{n=0}^{\infty} a_n e_*^{niw} :_{\tau}$ made by convergent power series $z^{\ell} \sum_{n=0}^{\infty} a_n z^n$. We call \mathfrak{H}_+ the half-series algebra. Its fundamental property is

Theorem 2.4 $(\mathfrak{H}_+, *_\tau)$ is a topological field consisting of 2π -periodic entire functions of w .

Proof is completed by showing the uniqueness of the inverse. It is reduced to show that

$$\sum_{n=0}^{\infty} a_n e_*^{niw} :_{\tau} *_\tau \sum_{k=0}^{\infty} b_k e_*^{kiw} :_{\tau} = 0$$

and $a_0 \neq 0$ gives $\sum_{k=0}^{\infty} b_k e_*^{kiw} :_{\tau} = 0$. The repeated use of Proposition 1.1 gives all $b_n = 0$. \square

2.2.2 Euler numbers

Recall the the generating function of Euler numbers

$$\frac{2}{e^z + e^{-z}} = \frac{e^z}{1 + e^{2z}} + \frac{e^{-z}}{1 + e^{-2z}} = \sum_{n=0}^{\infty} E_{2n} \frac{1}{(2n)!} z^{2n}, \quad |z| < \pi.$$

The l.h.s. is a convergent power series obtained by the method of indeterminate constants. Hence, by Proposition ?? gives

$$(2.9) \quad e_*^{e^{iw}} * \left(1 + \sum_{k=0}^{\infty} 2^k e_*^{kiw} \frac{1}{k!}\right)^{-1} + e_*^{-e^{iw}} * \left(1 + \sum_{k=0}^{\infty} (-2)^k e_*^{kiw} \frac{1}{k!}\right)^{-1} = \sum_{n=0}^{\infty} E_{2n} \frac{1}{(2n)!} e_*^{2niw},$$

where $e_*^{\pm e^{iw}} = \sum_{\ell=0}^{\infty} \frac{(\pm 1)^{\ell}}{\ell!} e_*^{\ell iw}$. Note that this identity holds for every expression τ such that $\text{Re } \tau > 0$.

On the other hand, by using the formal power series of $(iw)_*$, we can compute the inverses $\left(1 + \sum_{k=0}^{\infty} \frac{(2iw)_*^k}{k!}\right)^{-1}$, $\left(1 + \sum_{k=0}^{\infty} \frac{(-2iw)_*^k}{k!}\right)^{-1}$ by the method of indeterminate constants. Hence, we have also

$$(2.10) \quad e_*^{iw} * \left(1 + \sum_{k=0}^{\infty} (2iw)_*^k \frac{1}{k!}\right)^{-1} + e_*^{-iw} * \left(1 + \sum_{k=0}^{\infty} (-2iw)_*^k \frac{1}{k!}\right)^{-1} = \sum_{n=0}^{\infty} E_{2n} \frac{1}{(2n)!} (iw)_*^{2n}.$$

where r.h.s. is a formal power series of iw in the $*$ -product. It is clear that replacing $(iw)_*^k$ by e_*^{kiw} in (2.10) gives (2.9).

It is very interesting to compare the l.h.s. with $e_*^{iw}*(1+e_*^{2iw})_{*+}^{-1}+e_*^{-iw}*(1+e_*^{-2iw})_{*+}^{-1}$. The τ -expression of this is an entire function and its Taylor expansion is given by

$$(2.11) \quad :e_*^{iw}*(1+e_*^{2iw})_{*+}^{-1}+e_*^{-iw}*(1+e_*^{-2iw})_{*+}^{-1}:_{\tau} = 2 \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n e^{-\frac{(2n+1)^2}{4}\tau} (2n+1)^{2\ell} \frac{1}{(2\ell)!} (iw)^{2\ell}.$$

However, this is a 2π -periodic function, while $\sum_{n=0}^{\infty} E_{2n} \frac{1}{(2n)!} (iw)_*^{2n}$ is not 2π -periodic in the τ -expression. Note also that the l.h.s. of (2.10) may be computed as a 2π -periodic function, but the method of indeterminate constants ignores the periodicity. Therefore, we have to rewrite (2.10) within 2π -periodic functions to compare this with $e_*^{iw}*(1+e_*^{2iw})_{*+}^{-1}+e_*^{-iw}*(1+e_*^{-2iw})_{*+}^{-1}$.

Note that for every integer m , we have

$$\begin{aligned} e_*^{iw} * \left(1 + \sum_{k=0}^{\infty} (2i(w+2\pi m))_*^k \frac{1}{k!} \right)^{-1} &+ e_*^{-iw} * \left(1 + \sum_{k=0}^{\infty} (-2i(w+2\pi m))_*^k \frac{1}{k!} \right)^{-1} \\ &= \sum_{n=0}^{\infty} E_{2n} \frac{1}{(2n)!} (i(w+2\pi m))_*^{2n}. \end{aligned}$$

We will come back again to this problem in the forthcoming paper.

2.2.3 Bernoulli numbers

Recall the generating function of Bernoulli numbers:

$$z \left(\frac{1}{2} + \frac{1}{e^z - 1} \right) = \frac{z}{2} \left(\frac{1}{e^z - 1} - \frac{1}{e^{-z} - 1} \right) = \sum_{n=0}^{\infty} B_{2n} \frac{1}{(2n)!} z^{2n}.$$

Since $\frac{z}{e^z - 1}$ and $\frac{-z}{e^{-z} - 1}$ are computed by the method of indeterminate constants as

$$\left(\sum_n \frac{z^n}{(n+1)!} \right)^{-1} = \sum B_{2n} \frac{1}{(2n)!} z^{2n} - \frac{1}{2} z, \quad \left(\sum_n \frac{(-z)^n}{(n+1)!} \right)^{-1} = \sum B_{2n} \frac{1}{(2n)!} z^{2n} + \frac{1}{2} z,$$

these must hold in the $*$ -product by replacing z by e_*^{iw} . Hence, we have

$$(2.12) \quad \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(e_*^{iw})^n}{(n+1)!} \right)^{-1} + \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(-e_*^{iw})^n}{(n+1)!} \right)^{-1} = \sum_{n=0}^{\infty} B_{2n} \frac{1}{(2n)!} (e_*^{iw})^{2n}$$

The uniqueness of inverses is ensured by Theorem 2.4.

On the other hand, replacing z by iw and using the formal power series of $(iw)_*^n$, we have also the identity

$$(2.13) \quad \frac{1}{2} \left(\sum_n \frac{(iw)_*^n}{(n+1)!} \right)^{-1} + \frac{1}{2} \left(\sum_n \frac{(-iw)_*^n}{(n+1)!} \right)^{-1} = \sum_{n=0}^{\infty} B_{2n} \frac{(iw)_*^{2n}}{(2n)!}$$

where the r.h.s. is a formal power series of $(iw)_*^n$. This holds in every expression parameter τ , and the r.h.s. may be regarded as a formal power series of $:(iw)_*^{2n}:\tau$.

As in (2.10) it is very interesting to compare this with $iw_*(e_*^{iw}-1)_{*+}^{-1}-iw_*(e_*^{-iw}-1)_{*+}^{-1}$ under the condition $\text{Re } \tau > 0$. However, this is written in the form $iw_*g_*(w)$ by using a 2π -periodic function $g_*(w) = g_*(w+2\pi)$, while the r.h.s. of (2.13) has no such property. Hence, we have to rewrite the l.h.s. of (2.13) may be viewed as $\frac{iw}{2} * \left((e_*^{iw}-1)^{-1} - (e_*^{-iw}-1)^{-1} \right)$.

At this moment, we do not have an effective method to compare these, but we will come back again to this interesting problem in the forthcoming paper.

3 Fourier transform of tempered distributions

We first recall the definition of rapidly decreasing functions of several variables.

Definition 3.1 *A C^∞ -function, f on \mathbb{R}^n , is called a rapidly decreasing function if*

$$\sup_{x \in \mathbb{R}^n} |p(x) \partial^\alpha f(x)| < \infty$$

for every polynomial $p(x)$ and for every multi-index α .

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all rapidly decreasing functions. $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space under the family of seminorms

$$\|f\|_{k,m} = \sum_{|\alpha| \leq m} \sup (1 + |x|^2)^k |\partial^\alpha f|$$

First, express this space as the projective limit space of a family of Hilbert spaces. Define a family of inner products on $\mathcal{S}(\mathbb{R}^n)$ as follows:

$$\langle f, g \rangle_k = \sum_{|\alpha|+|\beta| \leq k} \int x^{2\beta} \partial^\alpha f \partial^\alpha g \, \vec{d}x$$

Make the topological completion of $\mathcal{S}(\mathbb{R}^n)$ using the norm topology defined by the inner product, and denote it by $\mathcal{S}^k(\mathbb{R}^n)$. The Sobolev lemma gives that $\mathcal{S}(\mathbb{R}^n) = \bigcap_k \mathcal{S}^k(\mathbb{R}^n)$.

Fourier transform is defined as follows:

$$\mathfrak{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\xi} \, \vec{d}x$$

where $\vec{d}x = \frac{1}{\sqrt{2\pi}} dx$. $\mathfrak{F}(f)(\xi)$ is sometimes denoted by $\hat{f}(\xi)$. Fourier transform is defined for L^1 -functions at first, and extends in various ways.

Lemma 3.1 *If f, f', f'' are continuous and summable, then $\hat{f}(\xi)$ is summable, and it holds that $\|f\|_{L^2} = \|\mathfrak{F}(f)\|_{L^2}$.*

Fundamental properties of Fourier transform are as follows:

$$\mathfrak{F}(\partial^\alpha f) = (-i\xi)^\alpha \mathfrak{F}(f), \quad \mathfrak{F}((ix)^\alpha f) = \partial^\alpha \mathfrak{F}(f).$$

These are all proved by integration by parts.

Fourier transform exchanges differentiation and multiplication by generators. This observation suggests that the most convenient topology for the function space on which Fourier transform is defined is the topology where multiplication and differentiation by coordinate functions are treated with the same weight.

It is well known that the Fourier transform \mathfrak{F} gives a topological isomorphism of $\mathcal{S}(\mathbb{R}_x^n)$ onto $\mathcal{S}(\mathbb{R}_\xi^n)$. Furthermore, it gives a topological isomorphism of $\mathcal{S}^k(\mathbb{R}_x^n)$ onto $\mathcal{S}^k(\mathbb{R}_\xi^n)$ for every k . Setting $\mathcal{S}(\mathbb{R}^n) = \bigcap_k \mathcal{S}^k(\mathbb{R}^n)$, it gives an isomorphism of $\mathcal{S}(\mathbb{R}^n)$.

Thus, \mathfrak{F} gives a topological isomorphism of the dual space $\mathcal{S}'(\mathbb{R}_\xi^n)$ onto $\mathcal{S}'(\mathbb{R}_x^n)$.

Let $\mathcal{S}^{-k}(\mathbb{R}_x^n)$ be the dual space of $\mathcal{S}^k(\mathbb{R}_x^n)$. The dual space of a Hilbert space is a Hilbert space by virtue of the Riesz theorem. Hence, $\mathcal{S}^{-k}(\mathbb{R}_x^n)$ is a Hilbert space. We see easily that

$$\mathcal{S}^{-\infty}(\mathbb{R}_x^n) = \bigcup_k \mathcal{S}^{-k}(\mathbb{R}_x^n)$$

with the inductive limit topology.

Lemma 3.2 *If $\operatorname{Re} \tau > 0$, then $:\delta_*(z+w):_\tau = \int_{\mathbb{R}} e^{itz} e^{itw - \frac{\tau}{4}t^2} dt$ is an entire function of $z + w$.*

Elements of $\mathcal{S}^{-\infty}(\mathbb{R}_x^n)$ are called **tempered distributions**. If a tempered distribution f is a function, that is, the value $f(x)$ is defined for every x , then f is called a **slowly increasing function**.

Lemma 3.3 *For a polynomial $p(x)$ with real coefficients, $e^{ip(x)}$ is a slowly increasing function.*

By Riesz's theorem \mathfrak{F} extends to a linear isometry of $\mathcal{S}^{-k}(\mathbb{R}_x^n)$ onto $\mathcal{S}^{-k}(\mathbb{R}_\xi^n)$. Hence, the Fourier transform \mathfrak{F} extends to a linear isomorphism of $\mathcal{S}^{-\infty}(\mathbb{R}_x^n)$ onto $\mathcal{S}^{-\infty}(\mathbb{R}_\xi^n)$.

Recall that if $\operatorname{Re} \tau > 0$, then $:\delta_*(x-w):_\tau = \frac{1}{\sqrt{\pi\tau}} e^{-\frac{1}{\tau}(x-w)^2}$ is rapidly decreasing. Suppose $f(x)$ is $e^{|x|^\alpha}$ -growth on \mathbb{R} with $0 < \alpha < 2$. Then the integral $\int f(x) : \delta_*(x-w) :_\tau dx$ is well-defined to give an entire function w.r.t. w .

The next theorem is the main tool to extend the class of $*$ -functions via Fourier transform.

Theorem 3.1 *Suppose $\operatorname{Re} \tau > 0$. For every tempered distribution $f(x)$, the τ -expression of*

$$\int_{-\infty}^{\infty} f(x) \delta_*(x-w) dx$$

is an entire function of w . In particular we see $\delta_(a-w) = \int_{-\infty}^{\infty} \delta(x-a) \delta_*(x-w) dx$.*

Proof The τ -expression is $\int_{-\infty}^{\infty} f(t)e^{-\frac{\tau}{4}t^2+itw}dt$. By restricting w to an arbitrary compact subset of \mathbb{C} , $e^{-\frac{\tau}{4}t^2+itw}$ is rapidly decreasing w.r.t. t . Hence, the integral $\int_{-\infty}^{\infty} f(t)e^{-\tau t^2+itw}dt$ exists for every tempered distribution $f(t)$. Since the complex differentiation ∂_w does not suffer the convergence, we see that this is holomorphic on the whole plane. \square

For every tempered distribution $f(x)$, we define a $*$ -function $f_*(w)$ by

$$(3.1) \quad f_*(w) = \int_{-\infty}^{\infty} f(x)\delta_*(x-w)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{f}(t)e_*^{-itw}dt.$$

where $\check{f}(t)$ is the inverse Fourier transform of $f(x)$. As $f(x)$ is a tempered distribution, one may write

$$\int f(x):\delta_*(x-w):_{\tau}dx = \frac{1}{2\pi} \iint f(x)e^{itx}:e_*^{-itw}:_{\tau}dtdx$$

under the existence of a rapidly decreasing function $:e_*^{-itw}:_{\tau}$ in the integrand. By the definition of Fourier transform of tempered distribution, one may exchange the order of integrations. Hence, letting $\check{f}(t)$ be the inverse Fourier transform of $f(x)$, we have

$$(3.2) \quad :\int_{\mathbb{R}} f(x)\delta_*(x-w)dx:_{\tau} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \check{f}(t):e_*^{-itw}:_{\tau}dt = :f_*(w):_{\tau}.$$

If another $*$ -function is given by $g_*(w) = \int g(x)\delta_*(x-w)dx$, we define the product by

$$(3.3) \quad f_*(w)*g_*(w) = \int_{-\infty}^{\infty} f(w)g(w)\delta_*(x-w)dx = \frac{1}{\sqrt{2\pi}} \int \left(\frac{1}{\sqrt{2\pi}} \int \check{f}(t-\sigma)\check{g}(\sigma)d\sigma \right) e_*^{-itw}dt,$$

if $f(x)g(x)$ is defined as a tempered distribution or the convolution product

$$(\check{f} \bullet \check{g})(t) = \frac{1}{\sqrt{2\pi}} \int \check{f}(t-\sigma)\check{g}(\sigma)d\sigma$$

is defined as a tempered distribution. Hence, (3.3) may be viewed as an integral representation of the intertwiner $I_0^{\tau}(f(x)) = f_*(w)$. If $f(x)$ is a slowly increasing function, applying (3.3) to the case $g_*(w) = \delta_*(a-w)$ gives

$$(3.4) \quad f_*(w) * \delta_*(a-w) = \int f(x)\delta(a-x)\delta_*(x-w)dx = f(a)\delta_*(a-w).$$

3.1 Several applications

Note that $\frac{1}{a-x}$, $a \notin \mathbb{R}$, is a slowly increasing function. If $\text{Im } a < 0$, then $\frac{1}{a-x} = \int_{-\infty}^0 e^{ita}e^{-itx}dt$. Regarding this as the Fourier transform of $\chi_{-}(t)e^{ita}$ supported on the negative half-line, we have $\int \frac{1}{a-x}\delta_*(x-w)dx = \int_{-\infty}^0 e^{ita}e_*^{-itw}dt = (a-w)_{*+}^{-1}$. It is not hard to verify

$$\int \frac{1}{a-x}\delta_*(x-w)dx = \begin{cases} (a-w)_{*+}^{-1} & \text{Im } a < 0 \\ (a-w)_{*-}^{-1} & \text{Im } a > 0 \end{cases}, \quad \text{Re } \tau > 0.$$

Although the product $\delta_*(x-w)*\delta_*(x-w)$ diverges, the next one is important

$$(3.5) \quad \delta_*(x-w)*\delta_*(x'-w) = \delta(x-x')\delta_*(x'-w)$$

in the sense of distribution. This is proved directly as follows:

$$\begin{aligned} \delta_*(x-w)*\delta_*(x'-w) &= \left(\frac{1}{2\pi}\right)^2 \iint e_*^{it(x-w)} * e_*^{is(x'-w)} dt ds \\ &= \left(\frac{1}{2\pi}\right)^2 \iint e^{itx+isx'} e_*^{-i(t+s)w} dt ds = \left(\frac{1}{2\pi}\right)^2 \iint e^{is(x'-x)} e_*^{i\sigma(x-w)} ds d\sigma = \delta(x'-x)\delta_*(x-w). \end{aligned}$$

Note the trick that the computation is done without showing the expression parameters. It is important to confirm this is true in the ordinary calculation, and we do not use the operator valued distributions.

For $\text{Re } \tau > 0$, we define

$$Y_*(w) = \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\infty} \delta_*(x-w) dx, \quad Y_*(-w) = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{-\varepsilon} \delta_*(x-w) dx.$$

It is clear that

$$\partial_w \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\infty} \delta_*(x-w) dx = - \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\infty} \partial_x \delta_*(x-w) dx = \delta_*(-w) = \delta_*(w).$$

By using (3.5) we have

$$Y_*(w)*Y_*(w) = Y_*(w), \quad Y_*(w)*Y_*(-w) = 0, \quad Y_*(w)+Y_*(-w) = \int_{\mathbb{R}} \delta_*(x-w) dx = 1.$$

We define

$$\text{sgn}_*(w) = Y_*(w) - Y_*(-w).$$

It is easy to see that $\text{sgn}_*(w)*\text{sgn}_*(w) = Y_*(w)+Y_*(-w) = 1$, $\text{sgn}_*(w)+\text{sgn}_*(-w) = 0$.

Since $\delta_*(z-w)$ is holomorphic in z , Cauchy integral theorem gives that every contour integral vanishes, but we see easily for every simple closed curve C

$$\frac{1}{2\pi i} \int_C \frac{1}{z} \delta_*(z-w) dz = \delta_*(w), \quad \text{Re } \tau > 0.$$

Note that $\text{v.p.} \frac{1}{x}$, $\text{Pf.} x^{-m}$, $m \in \mathbb{N}$ are tempered distribution which are not functions, but their Fourier transform may be viewed as slowly increasing functions. Hence we see

$$\begin{aligned} \text{v.p.} \int_{\mathbb{R}} \frac{1}{x} \delta_*(x-w) dx &= \frac{-i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{\frac{\pi}{2}} \text{sgn}(t) e_*^{-itw} dt = \frac{1}{2} (w_{*+}^{-1} + w_{*-}^{-1}) \\ \text{Pf.} \int_{\mathbb{R}} x^{-m} \delta_*(x-w) dx &= \frac{-i}{2} \int_{\mathbb{R}} \frac{1}{(m-1)!} (-it)^{m-1} \text{sgn}(t) e_*^{-itw} dt = (-1)^{m-1} \frac{1}{2} (w_{*+}^{-m} + w_{*-}^{-m}). \end{aligned}$$

3.1.1 Periodic distributions

A tempered distribution $f(x)$ is called a 2π -periodic tempered distribution, if $f(x)$ satisfies $f(x+2\pi) = f(x)$. For every distribution $f(x)$ of compact support, the infinite sum $\sum_n f(x+2\pi n)$ is a 2π -periodic tempered distribution. This procedure will be called the 2π -periodization.

As $f(t) = \sum_n e_*^{i(n+t)(a+w)}$ is a periodic function of period 1, Fourier expansion formula gives $f(t) = \sum_m (\int_0^1 f(s) e^{-2\pi i m s} ds) e^{2\pi i m t}$ and

$$f(0) = \sum_m \left(\int_0^1 \sum_n e_*^{i(n+s)(a+w)} e^{-2\pi i m s} ds \right) = \int_0^1 \sum_m \sum_n e_*^{i(n+s)(a+w)} e^{-2\pi i m s} ds$$

As $e_*^{i(n+s)(a+w)} e^{-i2\pi m s} = e_*^{i(n+s)(a+w)} e^{-i(2\pi m(n+s))}$,

$$\int_0^1 \sum_m \sum_n e_*^{i(n+s)(a+w)} e^{-2\pi i m s} ds = \sum_m \int_{\mathbb{R}} e_*^{is(a+w+2\pi m)} ds = \sum_m \delta_*(a+w+2\pi m).$$

It follows the fundamental relation between 2π -periodic tempered distributions and Fourier series

$$(3.6) \quad \sum_n \delta_*(a+2\pi n+w) = \sum_n e_*^{in(a+w)}.$$

A continuous function $f(x)$ on $[-\pi, \pi]$ extends to a (not continuous) 2π -periodic function $\tilde{f}_\pi(x)$ to give a 2π -periodic tempered distribution, where

$$\tilde{f}_\pi(x) = \frac{1}{2\pi} \sum_n \left(\int_{-\pi}^{\pi} f(s) e^{-ins} ds \right) e^{inx} = \sum_n a_n e^{inx}$$

Hence,

$$(3.7) \quad \tilde{f}_{\pi*}(w) = \int_{\mathbb{R}} \tilde{f}_\pi(x) \delta_*(x-w) dx = \sum a_n e_*^{inw}.$$

In particular by using $\frac{1}{in}(x^m e^{-inx})' = \frac{1}{in} m x^{m-1} e^{-inx} - x^m e^{-inx}$, we have

$$\tilde{x}_{\pi*}(w) = \int_{\mathbb{R}} \tilde{x}_\pi(x) \delta_*(x-w) dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{i}{n} (-1)^{n-1} e_*^{inw}.$$

$$\tilde{x}_{\pi*}^2(w) = \int_{\mathbb{R}} \tilde{x}_\pi^2(x) \delta_*(x-w) dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{2}{n^2} (-1)^{n-1} e_*^{inw}$$

Lemma 3.4 *If a periodic distribution $f_*(w)$ satisfies $\tilde{x}_{\pi*}(w)*f_*(w) = 0$, then*

$$f_*(w) = c \sum_n \delta_*(w+2\pi n) = c \sum_n e_*^{inw}.$$

Proof Note that $\tilde{x}_{\pi*}(w)*f_*(w) = \int \tilde{x}_\pi(x)f(x)\delta_*(x-w)dx$. Hence, $\tilde{x}_\pi(x)f(x) = 0$ where $f(x)$ is a periodic distribution. As $\tilde{x}_\pi(x)$ is a 2π -periodization of x , this means $xf(x) = 0$ on a neighborhood of 0. This gives $f(x) = c\delta(x)$ in a neighborhood of 0. The periodicity of $f(x)$ gives $f(x) = \sum_k \delta(x+2\pi k)$ and hence, $f_*(w) = \sum_k \delta_*(w+2\pi k)$. \square

3.2 Double-valued parallel sections

Proposition 1.5 shows that the equation $:e_*^{2iw}:\tau * f(w) = 1$ has the unique solution e_*^{-2iw} . On the other hand, two inverses $(a+w)_{*\pm}^{-1}$ are defined for every $a \in \mathbb{C}$ by the integral

$$:(a+w)_{*+}^{-1}:\tau = i \int_{-\infty}^0 :e_*^{it(a+w)}:\tau dt, \quad :(a+w)_{*-}^{-1}:\tau = -i \int_0^{\infty} :e_*^{it(a+w)}:\tau dt, \quad \text{Re } \tau > 0.$$

So far, the expression parameter τ is mainly restricted to the domain $\text{Re } \tau > 0$. However, for every $\tau \neq 0$ there is θ such that $e^{2i\theta}\tau > 0$. Keeping these in mind, we compute the τ -expression of

$$e^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w)} dt, \quad -e^{i\theta} \int_0^{\infty} e_*^{ite^{i\theta}(a+w)} dt.$$

For the first one, we see

$$(3.8) \quad :e^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w)} dt:\tau = e^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w) - \frac{1}{4}t^2 e^{2i\theta}\tau} dt$$

converges absolutely to give the inverse $a+w$. Moreover, we see by integration by parts

$$\begin{aligned} \frac{d}{d\theta} :e^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w)} dt:\tau &= :ie^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w)} dt:\tau + :e^{i\theta} \int_{-\infty}^0 \frac{d}{d\theta} e_*^{ite^{i\theta}(a+w)} dt:\tau \\ &= :ie^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w)} dt:\tau - :ie^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w)} dt:\tau = 0 \end{aligned}$$

whenever the property $\text{Re } e^{2i\theta}\tau > 0$ is retained. This is proved also by Cauchy's integration theorem via the rotation of the path of integration.

If the expression parameter is restricted in a simply connected domain D of $\mathbb{C} \setminus \{0\}$, one can treat $:(a+w)_{*\pm}^{-1}:\tau$ as a single valued parallel section on D . In such a restricted domain, products of inverse elements are given by the resolvent identities, which gives an associative algebra similar to the half-series algebra.

This observation suggests that the natural boundary of the expression parameter τ of the discrete summation such as $\theta_3(w, *)$ causes an essential difference between $\delta_*(w)$ and $\theta_3(w, *)$.

The following proposition shows that $:(a+w)_{*+}^{-1}:\tau$ and $:(a+w)_{*-}^{-1}:\tau$ are connected by a parallel translation along a closed curve of expression parameters. But since parallel translations in this case are always considered with θ such that $|2\theta+\sigma| < \frac{\pi}{2}$, we refer such translation as **joint parallel displacements/translations**.

Proposition 3.1 *Suppose $|2\theta+\sigma| < \frac{\pi}{2}$. Then, the $|\tau|e^{i\sigma}$ -expression of $e^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w)} dt$ is independent of θ . That is, in the domain (θ, σ) , the element given by the integral*

$$:e^{i\theta}(e^{i\theta}(a+w))_{*+}^{-1}:_{|\tau|e^{i\sigma}} = :e^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w)} dt:_{|\tau|e^{i\sigma}}$$

depends only on the expression parameter $\tau = |\tau|e^{i\sigma}$. Hence, it may be viewed as a single element. However, moving θ from 0 to $-\frac{\pi}{2}$ in this domain, we have

$$:i \int_{-\infty}^0 e_*^{it(a+w)} dt:_{\tau} = : \int_{-\infty}^0 e_*^{t(a+w)} dt:_{-\tau}.$$

Moving θ from 0 to π , we see

$$: \int_{-\infty}^0 e_*^{it(a+w)} dt:_{\tau} = : - \int_{-\infty}^0 e_*^{-it(a+w)} dt:_{\tau} = - : \int_0^{\infty} e_*^{it(a+w)} dt:_{\tau}.$$

In particular, the complex rotation $e^{i\theta}$ from $\theta = 0$ to π of the path of integration together with the expression parameter, exchanges $(a+w)_{+}^{-1}$ and $(a+w)_{*-}^{-1}$.*

Remark The last equality does not imply $(a+w)_{*+}^{-1} = (a+w)_{*-}^{-1}$. It expresses only the result of parallel translation along a closed path. For $\text{Re } \tau > 0$, these may be written as

$$(a+w)_{*+}^{-1} = i \int_{-\infty}^0 e_*^{it(a+w)} dt, \quad (a+w)_{*-}^{-1} = -i \int_0^{\infty} e_*^{it(a+w)} dt,$$

without showing the expression parameters explicitly.

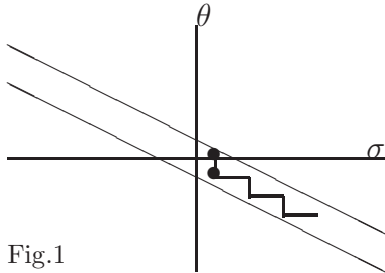


Fig.1

Since $:e^{i\theta} \int_{-\infty}^0 e_*^{ite^{i\theta}(a+w)} dt:_{|\tau|e^{i\sigma}}$ is viewed as various expressions of a single element defined on the domain $|2\theta+\sigma| < \frac{\pi}{2}$, it may be written by $(a+w)_{*}^{-1}$. However, this is a double-valued single parallel section. We do not use the notation $(a+w)_{*}^{-1}$ as this may cause some confusion. In our expression, $(a+w)_{*\pm}^{-1}$ are viewed as parallel sections defined on the space of expression parameters sitting in the domain $|2\theta+\sigma| < \frac{\pi}{2}$. Proposition 3.1 shows how these expressions are related to each other.

As $\delta_*(a+w) = (a+w)_{*+}^{-1} - (a+w)_{*-}^{-1}$, Proposition 3.1 must give that $e^{i\theta}\delta_*(e^{i\theta}(a+w))$ depends only on the expression parameter on the domain $|2\theta+\sigma| < \frac{\pi}{2}$. Indeed, we see

$$(3.9) \quad :e^{i\theta}\delta_*(e^{i\theta}(a+w)):_{|\tau|e^{i\sigma}} = \frac{1}{\sqrt{\pi|\tau|e^{i\sigma}}} e^{-\frac{1}{|\tau|e^{i\sigma}}(a+w)^2}$$

by the Fourier transform and this is rapidly decreasing. Setting $\sigma = -2\theta$ and moving θ from 0 to π , we see that

$$-:\delta_*(-(a+w)):_{\tau} = -\delta_*(a+w):_{\tau} = \delta_*(a+w):_{\tau}.$$

Proposition 3.2 *The mapping $\delta_* : \mathbb{C} \setminus \{0\} \rightarrow \text{Hol}(\mathbb{C})$ defined by $\tau \mapsto \delta_* :_{\tau}$ is a double-valued parallel section with \pm -ambiguity.*

In contrast with Proposition 1.5, we easily see the following:

Proposition 3.3 *For every $\tau \neq 0$, the real analytic solution of the differential equation*

$$(a+w)*_{\tau}f(w) = 0$$

is $f(w) = Ce^{-\frac{1}{\tau}(a+w)^2}$. That is, the space of solutions form the trivial bundle $\mathbb{C}_ \times \mathbb{C}e^{-\frac{1}{\tau}(a+w)^2}$, while $C:\delta_*(a+w):_{\tau} = \frac{1}{\sqrt{\pi\tau}}e^{-\frac{1}{\tau}w^2}$ forms a Möbius bundle over \mathbb{C}_* .*

Note $f(\theta, \sigma) = :e^{i\theta}\delta_*(e^{i\theta}(a+w)):_{|\tau|e^{i\sigma}}$ is a smooth function defined on the strip in Fig.1 such that $\partial_{\theta}f(\theta, \sigma) = 0$. But it may loose the important information to write this $f(\theta, \sigma) = f(\sigma)$. This implies that $\delta_*(a+w)$ must be treated as a parallel section of the Möbius bundle over \mathbb{C}_* , while the solution of $(a+w)*_{\tau}f(w) = 0$ can be treated as trivial parallel sections of the product bundle. This is not a minor difference, but gives a big philosophical difference. It is natural to expect there is a “more” twisted bundles. In the last part we give a candidate for this by considering $*$ -exponential function $e_*^{sw_*^{\ell}}$ for $\ell > 3$.

Since $\int_{-\infty}^0 e_*^{it(a+w)} dt$ and $-\int_0^{\infty} e_*^{it(a+w)} dt$ are connected by a parallel transform along a closed curve, these may be viewed as the same element. Though Proposition 3.1 insists these can never be distinguished globally, these two can be distinguished locally.

It is remarkable that there is a single object with many indistinguishable individuals.

In spite of this, it is easy to see by change of variables that for every real $\lambda \neq 0$,

$$(3.10) \quad (\lambda(a+w))_{*\pm}^{-1} = \lambda^{-1}(a+w)_{*\pm}^{-1}, \quad \delta_*(\lambda(a+w)) = \lambda^{-1}\delta_*(a+w), \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

For instance, replacing $t\lambda \rightarrow t'$ gives

$$i \int_{-\infty}^0 e_*^{it\lambda(a+w)} dt = i\lambda^{-1} \int_{-\infty}^0 e_*^{it'(a+w)} dt'.$$

On the other hand, we see that

$$(3.11) \quad \int_{\mathbb{R}} \delta_*(x+w) dx = \frac{1}{\sqrt{\pi\tau}} \int_{\mathbb{R}} e^{-\frac{1}{\tau}(x+w)^2} dx = \frac{1}{\sqrt{\pi\tau}} \int_{\mathbb{R}} e^{-\frac{1}{\tau}(x+w)^2} dw = 1, \quad \text{Re } \tau > 0,$$

is independent of the expression parameter τ .

3.3 Complex rotations of $*$ -functions

Recall that $\delta_*(a+w)$ is double-valued. In precise, $:e^{i\theta}\delta_*(e^{i\theta}(a+w)):_{{\tau|e^{i\sigma}}}$ is a double-valued parallel section under the joint parallel translation. Indeed, this is given by

$$:e^{i\theta}\delta_*(e^{i\theta}(a+w)):_{{\tau|e^{i\sigma}}} = \frac{1}{\sqrt{\pi|\tau|e^{i\sigma}}} e^{-\frac{1}{|\tau|e^{i\sigma}}(a+w)^2}$$

by the Fourier transform and it is rapidly decreasing on the domain $|2\theta+\sigma| < \frac{1}{2}\pi$. Note that the value dose not depend on θ . Hence, for every tempered distribution $f(x)$ the integral

$$:f_*(w):_{{\tau e^{i\sigma}}} = \int_{\mathbb{R}} f(x) :e^{i\theta}\delta_*(e^{i\theta}(x+w)):_{{\tau e^{i\sigma}}} dx$$

is defined as a double-valued parallel section with \pm -ambiguity on the domain $|2\theta+\sigma| < \frac{1}{2}\pi$.

An interesting phenomenon appears in the integral

$$(3.12) \quad \int_{\mathbb{R}} e^{isx^\ell} \delta_*(x-w) dx, \quad s \in \mathbb{R},$$

where ℓ is an integer $\ell \geq 3$. As e^{isx^ℓ} is slowly increasing function of x , the integral converges and defined an element which looks an element written as $e_*^{isw_*^\ell}$. We now consider the rotation $e^{i\theta}x$ of the path of integral from 0 to $2\pi/\ell$ by keeping $\text{Re } e^{2i\theta}\tau > 0$. Then the integral

$$\int_{\mathbb{R}} e^{is(e^{i\theta}x)^\ell} \delta_*(e^{i\theta}x-w) d(e^{i\theta}x)$$

changes into

$$e^{i\frac{2\pi}{\ell}} \int_{\mathbb{R}} e^{isx^\ell} \delta_*(x-w) dx.$$

Thus we see what is defined by (3.12) is not $e_*^{isw_*^\ell}$ but something like its ℓ -th root. Recalling that $e^{i\theta}\delta_*(e^{i\theta}x-w)$ is independent of θ , we see that it is impossible to distinguish

$$e^{i\frac{2\pi k}{\ell}} \int_{\mathbb{R}} e^{isx^\ell} \delta_*(x-w) dx, \quad k = 0, 1, 2, \dots, \ell-1.$$

The multi-valued nature makes it difficult to define

$$:f_*(w):_{\tau e^{i\sigma}} + :g_*(w):_{\tau e^{i\sigma}}$$

if the expression parameter can move independently. Thus it is better to define

$$:f_*(w) + g_*(w):_{\tau e^{i\sigma}} = \int_{\mathbb{R}} (f(x) + g(x)) :e^{i\theta} \delta_*(e^{i\theta}(x + w)) :_{\tau e^{i\sigma}} dx.$$

3.3.1 Double-valued nature disappears under integration

If $\operatorname{Re} \tau > 0$, we easily see that $\int_{\mathbb{R}} :\delta_*(a - w):_{\tau} dw = 1$, independent of the expression parameter τ . Hence, (3.4) shows also

$$\int_{\mathbb{R}} :f_*(w) * \delta_*(a - w):_{\tau} dw = f(a),$$

independent of the expression parameter τ . Similarly, we have

$$\frac{1}{e^{i\theta} \sqrt{\pi \tau}} \int_{\mathbb{R}} e^{-e^{-2i\theta} \frac{1}{\tau} (e^{i\theta} x + w)^2} d(e^{i\theta} x) = \frac{1}{\sqrt{\pi \tau}} \int_{\mathbb{R}} e^{-\frac{1}{\tau} (x + e^{-i\theta} w)^2} dx = 1.$$

Keeping this in mind, we define

$$(3.13) \quad \int_{\mathbb{R}} :f_*(w) * e^{i\theta} \delta_*(e^{i\theta}(x + w)) :_{\tau e^{i\sigma}} dw = f(a)$$

and call this the **formal *-integration**.

3.3.2 Another *-inverse of $\sin_* \pi w$.

A rotations of path of integration gives sometimes a powerful tool of calculation. Here we give an example, though this is not directly relevant to our purpose.

Recall that the classical formula

$$(3.14) \quad \sin \pi z \int_{-\infty}^{\infty} \frac{e^{tz}}{1 + e^t} dt = \pi = \Gamma(z) \Gamma(1 - z)$$

obtained easily by Cauchy's integral formula. The proof may be applied to our case to obtain an inverse of $\sin_* w$: Consider first the integral $(e_*^{2\pi i(z+w)} - 1) * \int_{-\infty}^{\infty} \frac{e_*^{t(z+w)}}{1 + e^t} dt$ for the case $\operatorname{Re} \tau < 0$. Including $(e_*^{2\pi i(z+w)} - 1)$ among the integrand, we have integrals along $2\pi i + \mathbb{R}$ to positive direction and along \mathbb{R} to the negative direction. Hence, by Cauchy integration formula, we have the negative residue:

$$(e_*^{2\pi i(z+w)} - 1) * \int_{-\infty}^{\infty} \frac{e_*^{t(z+w)}}{1 + e^t} dt = 2\pi i e_*^{\pi i(z+w)}, \quad (\operatorname{Re} \tau < 0).$$

Multiplying $e_*^{-\pi i(z+w)}$ to both sides by using the associativity (1.10), we have

$$2i \sin_* \pi(z+w) * \int_{-\infty}^{\infty} \frac{e_*^{t(z+w)}}{1+e^t} dt = 2\pi i$$

This gives another $*$ -inverse of $\sin_* \pi(z+w)$ different from $(\sin_* \pi(z+w))_{*\pm}^{-1}$ given by the theta function, since the latter is defined for $\operatorname{Re} \tau > 0$.

To investigate the relation between these inverses, we have to rotate the expression parameter. Note that Cauchy's integration theorem gives for every $-\pi < a < \pi$

$$f_*(w) = \int_{-\infty}^{\infty} \frac{e_*^{tw}}{1+e^t} dt = \int_{-\infty}^{\infty} \frac{e_*^{(t+ia)w}}{1+e^{t+ia}} dt = \int_{-\infty}^{\infty} \frac{e_*^{tw}}{e^{-ia}+e^t} dt * e_*^{ia(w-1)}.$$

The r.h.s. does not depend on a .

Fix τ so that $\operatorname{Re} \tau < 0$. Consider $e^{i\sigma} \tau$ -expression instead of τ -expression together with the $e^{i\theta}$ -rotation of the path of integral by keeping $\operatorname{Re} e^{i(2\theta+\sigma)} \tau < 0$, and set

$$:g_{*\theta}(w):_{e^{i\sigma}\tau} = : \int_{-\infty}^{\infty} \frac{e_*^{e^{i\theta}tw}}{e^{-ia}+e^{e^{i\theta}t}} d(e^{i\theta}t) :_{e^{i\sigma}\tau}, \quad \operatorname{Re} e^{2i\theta} e^{i\sigma} \tau < 0.$$

Since there is no singular point in a small sector, the rotation of the path changes nothing. This is also confirmed by checking

$$\frac{d}{d\theta} \int_{-\infty}^{\infty} \frac{e_*^{e^{i\theta}tw}}{e^{-ia}+e^{e^{i\theta}t}} d(e^{i\theta}t) = 0$$

via integration by parts. We set $g_{*\theta}(w) * e_*^{ia(w-1)} = f_{*\theta}(w)$. Since $f(w) = f_{*\theta}(w) = g_{*\theta}(w) * e_*^{ia(w-1)}$, we also have the identities

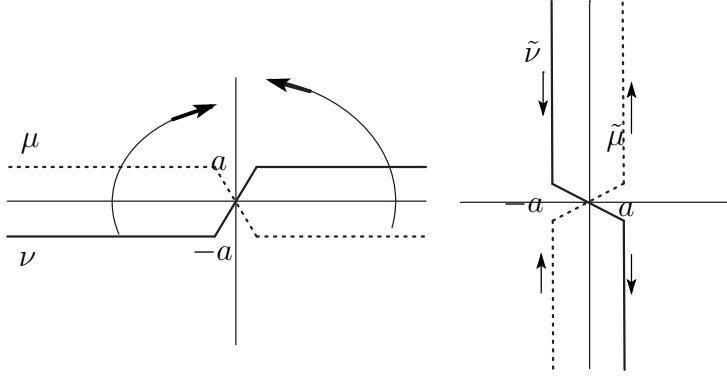
$$f_{*\theta}(w) = f_{*\theta}(1-w), \quad f_{*\theta}(w) + f_{*\theta}(w+1) = \int_{-\infty}^{\infty} e_*^{e^{i\theta}tw} d(e^{i\theta}t) = e^{i\theta} \delta_*(-e^{i\theta}w).$$

Hence, we have

$$(3.15) \quad \frac{1}{\pi} \sin_* \pi w * \int_{-\infty}^{\infty} \frac{e_*^{e^{i\theta}tw}}{1+e^{e^{i\theta}t}} d(e^{i\theta}t) = 1.$$

Taking the path of integration as in μ and ν in the l.h.s. figure below. Then, it is easy to see

$$2f_*(w) = \int_{\mu} \frac{e_*^{(t+ia)w}}{1+e^{t+ia}} dt + \int_{\nu} \frac{e_*^{(t-ia)w}}{1+e^{t-ia}} dt.$$



Set $\theta = -\frac{1}{2}\sigma$ and move σ from 0 to 2π in the path μ , and 0 to -2π in the path ν . At $\sigma = \pi, -\pi$, the expression parameter is $-\tau$ and the path of integrals are changed as $\tilde{\mu}$ and $\tilde{\nu}$ avoiding singular points. Thus, $2f_*(w)$ is changed into :

$$:f_{*-\frac{\pi}{2}}(w):_{-\tau} + :f_{*\frac{\pi}{2}}(w):_{-\tau} = : \int_{\tilde{\mu}} \frac{e^{-itw}}{1+e^{-it}} d(-it) :_{-\tau} + : \int_{\tilde{\nu}} \frac{e^{itw}}{1+e^{it}} d(it) :_{-\tau}.$$

Note that the alternating periodicity $w \rightarrow w+1$ appears in the r.h.s. because the paths of integrations are reversed in the first and second term of the r.h.s.

Switch the path of integration into upper and lower circuits. We now count the residues in the upper circuits and lower circuits separately. Then

$$:f_{*-\frac{\pi}{2}}(w):_{-\tau} + :f_{*\frac{\pi}{2}}(w):_{-\tau} = 2\pi \sum_{n=0}^{\infty} :e_*^{\pi i(2n+1)w} :_{-\tau} - 2\pi \sum_{n=0}^{\infty} :e_*^{-\pi i(2n+1)w} :_{-\tau}.$$

Thus, we see the following:

Theorem 3.2 *By altering the path of integration of l.h.s. by using the double-valued nature of $\delta_*(w)$ so that it obtains the alternating periodicity $w \rightarrow w+1$, we have*

$$(3.16) \quad 2 \int_{-\infty}^{\infty} \frac{e_*^{tw}}{1+e^t} dt = \pi(\sin_{*+}^{-1} \pi w + \sin_{*-}^{-1} \pi w).$$

In general, it is dangerous to express such identity without showing the expression parameters. But, it should be permitted when the identity is obtained via several rotations of expression parameters.

We often use rotations of path of integration together with rotations of expression parameters to observe the branching/periodic behavior of the object.

In the later chapter, we will show that elements which changes sign under the joint parallel displacement along a closed curve have a similar properties that are called fermion in physics.

The notion of *parity* is an important notion in quantum physics. This is explained as the statistical characters of particles. However, such objects are treated in usual calculus. Therefore, one can ask as a mathematical question

“Is such a notion mathematically consistent ?”

We think this question must be proved mathematically. That means such a notion should be defined mathematically in ordinary calculus, just as the geometric notions of “length”, “area” or “volume”.

3.4 Star-exponential functions of order < 2

Let $h(x)$ be a slowly increasing smooth function on \mathbb{R} of growth order α , $\alpha < 2$, such as $\sqrt{1+x^2}$ or $\frac{x^3}{x^2+1}$. Formula (1.21) gives that if $\text{Re}\tau > 0$, then for every $s \in \mathbb{C}$, $e^{sh(x)}:\delta_*(x-w):_\tau$ is rapidly decreasing w.r.t. x . Hence, the integral

$$(3.17) \quad \int_{\mathbb{R}} e^{sh(x)}:\delta_*(x-w):_\tau dx$$

is well-defined.

Theorem 3.3 *If $\text{Re}\tau > 0$, $\int_{\mathbb{R}} e^{sh(x)}:\delta_*(x-w):_\tau dx$ satisfies the exponential law w.r.t. s . Moreover, this satisfies the evolution equation*

$$\frac{d}{ds} f_s(w) = :h_*(w):_\tau *_\tau f_s(w), \quad f_0(w) = 1,$$

where $h_*(w) = \int h(x)\delta_*(x-w)dx$. Hence, this gives a complex one parameter group. Thus, we denote this integral (3.17) by $e_*^{sh_*(w)}$, $s \in \mathbb{C}$.

Proof Differentiating (3.17) by s , we have

$$\int_{\mathbb{R}} h(x)e^{sh(x)}\delta_*(x-w)dx.$$

Since $h_*(w) = \int h(x)\delta_*(x-w)dx$, we have

$$h_*(w) * \int_{\mathbb{R}} e^{sh(x)}:\delta_*(x-w):_\tau dx = \iint h(x')e^{sh(x')}\delta_*(x'-w)*\delta_*(x-w):_\tau dx'dx$$

Since (3.5) gives $\delta_*(x'-w)*\delta_*(x-w) = \delta(x-x')\delta_*(x-w)$ in the sense of distribution, this is

$$\int h(x)e^{sh(x)}:\delta_*(x-w):_\tau dx = \partial_s \int_{\mathbb{R}} h(x)e^{sh(x)}\delta_*(x-w)dx$$

It follows

$$\frac{d}{ds} e_*^{sh_*(w)} = h_*(w) * e_*^{sh_*(w)}.$$

The exponential law is verified directly by the computations as follows:

$$e_*^{sh_*(w)} * e_*^{s'h_*(w)} = \iint e^{sh(x)+s'h(y)}\delta_*(x-w)*\delta_*(y-w)dx dy = \iint e^{sh(x)+s'h(x)}\delta_*(x-w)dx dy.$$

□

In the definition above, the rotation of expression parameters cannot be considered in general. However, if $h(x)$ is a rational function such as $\frac{x^3}{(x+1+i)(x+1-i)}$, certain complex rotations allowed together with rotations of expression parameters.

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{sh(e^{i\theta}x)} : \delta_*(e^{i\theta}x - w) :_{|\tau|e^{i\sigma}} d(e^{i\theta}x)$$

is well-defined for (θ, σ) such that $|-2\theta + \sigma| < \frac{\pi}{2}$, $e^{i\theta}(1 \pm i) \notin \mathbb{R}$, and this is independent of θ in this domain. Thus, the value of the integral has a discontinuous jump given by the residues at the singular points.

In such a case, one can consider the **joint parallel displacement** along a curve in the expression parameter space by setting $\theta = \frac{\sigma}{2}$.

3.5 Star-exponential function of w_*^2

As we have seen, the $*$ -exponential function $e_*^{sh_*(w)}$ is very naive for the order of $h(x)$ is less than 2. In this section, we treat the $*$ -exponential function of the quadratic form w_*^2 . Noting that $:w_*^2:_\tau = w^2 + \frac{\tau}{2}$ in the τ -expression, we now define the star-exponential function of w_*^2 by the real analytic solution of the evolution equation

$$(3.18) \quad \frac{d}{dt} f_t = :w_*^2:_\tau *_\tau f_t, \quad f_0 = 1.$$

Precisely, this is

$$\frac{d}{dt} f_t = \frac{\tau^2}{4} f_t'' + \tau w f_t' + (w^2 + \frac{\tau}{2}) f_t, \quad f_0 = 1.$$

To solve this, we set $:f_t:_\tau = g(t)e^{h(t)w^2}$, by using the uniqueness of real analytic solutions. Then, we have a system of the ordinary differential equations:

$$(3.19) \quad \begin{cases} \frac{d}{dt} h(t) = (1 + \tau h(t))^2, & h(0) = 0 \\ \frac{d}{dt} g(t) = \frac{1}{2}(\tau^2 h(t) + \tau)g(t), & g(0) = 1. \end{cases}$$

The solution $:e_*^{tw_*^2}:_\tau$ is given by

$$(3.20) \quad :e_*^{tw_*^2}:_\tau = \frac{1}{\sqrt{1-\tau t}} e^{\frac{t}{1-\tau t} w^2}, \text{ for } \forall \tau, t\tau \neq 1, \quad (\text{double-valued}).$$

Note that no restriction to τ . $e_*^{tw_*^2}$ is obtained for all τ . This solution is obtained also via the intertwiner $I_0^\tau e^{tw^2}$. (See (3.26).) The Weyl ordered expression ($\tau=0$ -expression) gives $:e_*^{tw_*^2}:_0 = e^{tw^2}$ without singular point, and the τ -expression is $:e_*^{tw_*^2}:_\tau = I_0^\tau(e^{tw^2})$, but note that these have singularity at $t\tau=1$.

It is rather surprising that the solution has a branching singular point and hence, this does not form a complex one parameter group whenever $\tau \neq 0$ is fixed. Moreover, the solution is double-valued w.r.t. the variable t .

Since

$$\lim_{t \rightarrow \infty} \frac{\sqrt{t}}{\sqrt{1-t\tau}} e^{\frac{t}{1-t\tau}} = \frac{1}{\sqrt{-\tau}} e^{-\frac{1}{\tau} w^2},$$

we see that $\frac{1}{\sqrt{-\tau}} e^{-\frac{1}{\tau} w^2}$ is invariant by $e_*^{sw_*^2}$, that is,

$$:e_*^{sw_*^2}:_\tau *_\tau \frac{1}{\sqrt{-\tau}} e^{-\frac{1}{\tau} w^2} = \frac{1}{\sqrt{-\tau}} e^{-\frac{1}{\tau} w^2}.$$

Double-valued nature of $e_*^{tw_*^2}$ makes it difficult to select the domain where $e_*^{sw_*^2} + e_*^{tw_*^2}$ is a single valued function of two independent variables (s, t) .

Moreover, it should be very careful to treat elements such as

$$\sinh_* sw_*^2 = \frac{1}{2}(e_*^{sw_*^2} - e_*^{-sw_*^2}), \quad \cosh_* sw_*^2 = \frac{1}{2}(e_*^{sw_*^2} + e_*^{-sw_*^2})$$

for these are not double-valued, but triple valued functions of s and these two have no difference. Since

$$:\sinh_* sw_*^2:_\tau = \frac{1}{2\sqrt{1-s\tau}} e^{\frac{s}{1-s\tau} w^2} - \frac{1}{2\sqrt{1+s\tau}} e^{-\frac{s}{1+s\tau} w^2}$$

there are two singular points $s = \pm\tau^{-1}$ and we can choose the sign $\sqrt{1-s\tau}$, $\sqrt{1+s\tau}$ independently. Hence, $:\sinh_* 0w_*^2:_\tau$ may be seen as $1, 0, -1$. Thus, we have to restrict the domain for s to treat an element such as

$$f(s) = \sum_n :a_n e_*^{c_n sw_*^2}:_\tau, \quad a_n, c_n \in \mathbb{C}$$

as a single valued function.

In spite of such a difficulty, if a continuous curve C does not hit singular points, then $:e_*^{tw_*^2}:_\tau$ can be treated as a continuous function on C . Hence, the uniqueness of real analytic solution gives the exponential law $e_*^{sw_*^2} *_\tau e_*^{tw_*^2} = e_*^{(s+t)w_*^2}$:

$$\frac{1}{\sqrt{1-\tau s}} e^{\frac{s}{1-\tau s} w^2} *_\tau \frac{1}{\sqrt{1-\tau t}} e^{\frac{t}{1-\tau t} w^2} = \frac{1}{\sqrt{1-\tau(s+t)}} e^{\frac{s+t}{1-\tau(s+t)} w^2}.$$

Indeed, this is obtained by solving (3.19) with initial data $h(0) = \frac{t}{1+\tau t}$, $g(0) = \frac{1}{\sqrt{1-t\tau}}$ combined with calculations such as $\sqrt{a}\sqrt{b} = \sqrt{ab}$, $\sqrt{a}/\sqrt{a} = \sqrt{1} = \pm 1$.

Similarly, we have the exponential law $e^s *_\tau e_*^{tw_*^2} = e_*^{s+tw_*^2}$ with the ordinary scalar exponential functions.

Recall that if a continuous curve C does not hit singular points, then $:e_*^{tw^2}:_\tau$ can be treated as a continuous function on C . Hence, one can treat the integral $\int_C :e_*^{tw^2}:_\tau dt$ without ambiguity. By this reason, it is better to define $\sinh_* sw_*^2$, $\cosh_* sw_*^2$ by the integral

$$(3.21) \quad \sinh_* sw_*^2 = \frac{1}{2} \int_{-s}^s w_*^2 e_*^{tw_*^2} dt, \quad \cosh_* sw_*^2 = \frac{1}{2} \frac{d}{ds} \int_{-s}^s e_*^{tw_*^2} dt$$

via operations in the continuous calculus.

Formula (3.20) is easily inverted by setting $\frac{t}{1-t\tau} = a$ to obtain

$$(3.22) \quad e^{aw^2} = : \frac{1}{\sqrt{1+a\tau}} e_*^{\frac{a}{1+a\tau} w_*^2} :_\tau.$$

Thus, this makes calculations of $*_\tau$ -product easy for exponential functions of quadratic forms by the exponential law:

$$(3.23) \quad e^{aw^2} *_\tau e^{bw^2} = \frac{1}{\sqrt{(1+a\tau)(1+b\tau)}} : e_*^{(\frac{a}{1+a\tau} + \frac{b}{1+b\tau}) w_*^2} :_\tau = \frac{1}{\sqrt{1-ab\tau^2}} e^{\frac{a+b+2ab\tau}{1-ab\tau^2} w^2}$$

So far, the expression parameter τ is fixed, but the double-valued nature of $:e_*^{tw^2}:_\tau$ appears in the parallel translation along expression parameters.

The next one may sound strange:

Proposition 3.4 *The identity 1 is connected to -1 by a parallel translation along a closed curve of expression parameters. However, the identity 1 here is not the absolute scalar but the identity element of the group $\{e_*^{tw^2}\}$.*

We can locally distinguish the \pm sign, but we cannot globally.

Note for a change of generator Take a new generator, \hat{w} to be $\hat{w} = aw$. Then, we see

$$(3.24) \quad :e_*^{t\hat{w}^2}:_\tau = \frac{1}{\sqrt{1-ta^2\tau}} e^{\frac{t}{1-ta^2\tau} a^2 w^2} = :e_*^{ta^2 w^2}:_\tau \quad c.f.(1.11).$$

Set $t = -s\frac{1}{a^2}$ and $\tau = -1$ to obtain

$$:e_*^{t\hat{w}^2}:_\tau = \frac{1}{\sqrt{1-ta^2\tau}} e^{\frac{t}{1-ta^2\tau} a^2 w^2} = :e_*^{-sw_*^2}:_\tau = :e_*^{-sw_*^2}:_\tau \quad c.f.(1.11).$$

Hence, every $:e_*^{tw^2}:_\tau$ is obtained from the generating function of the Laguerre polynomials.

3.5.1 The generating function of Laguerre polynomials

The generating function of Laguerre polynomials $L_n^{(\alpha)}(x)$ is given as follows:

$$\frac{1}{(1-t)^{\alpha+1}} e^{-\frac{t}{1-t}x} = \sum_{n \geq 0} L_n^{(\alpha)}(x) t^n, \quad (|t| < 1).$$

Differentiating both sides by x , we have also the relations

$$(3.25) \quad \frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x).$$

If $\alpha = -\frac{1}{2}$, this is the $\tau = -1$ expression of $e_*^{-tw_*^2}$, i.e.

$$:e_*^{-tw_*^2}:_{-1} = \frac{1}{(1-t)^{\frac{1}{2}}} e^{-\frac{t}{1-t}w^2} = \sum_{n \geq 0} L_n^{(-\frac{1}{2})}(w^2) t^n.$$

Setting $x = w^2$, we see

$$\frac{d}{dx} :e_*^{-tw_*^2}:_{-1} = \frac{1}{(1-t)^{\frac{1}{2}}} \frac{d}{dx} e^{-\frac{t}{1-t}x} = - \sum_{n \geq 0} L_n^{(\frac{1}{2})}(x) t^{n+1}.$$

Minding these, we define $*$ -Laguerre polynomials $L_n(w^2, \tau) = :L_n(w^2, *):_\tau$ by

$$e_*^{tw_*^2} = \sum_n L_n^{(-\frac{1}{2})}(w^2, *) \frac{1}{n!} t^n, \quad L_n^{(-\frac{1}{2})}(w^2, \tau) = \frac{d^n}{dt^n} \Big|_{t=0} \frac{1}{(1-t\tau)^{\frac{1}{2}}} e^{\frac{1}{\tau(1-t\tau)}w^2} e^{-\frac{1}{\tau}w^2}.$$

As $t = 0$ is a regular point, these are well-defined, and the exponential law gives

$$L_n^{(-\frac{1}{2})}(w^2, *) = \sum_{k+\ell=n} L_k^{(-\frac{1}{2})}(w^2, *) * L_\ell^{(-\frac{1}{2})}(w^2, *).$$

Note that setting $x = w^2$,

$$\frac{d}{dt} \frac{x^{\alpha-1}}{(1-t\tau)^\alpha} e^{\frac{1}{\tau(1-t\tau)}} = \frac{d}{dx} \frac{1}{\tau} \frac{x^\alpha}{(1-t\tau)^{\alpha+1}} e^{\frac{1}{\tau(1-t\tau)}}.$$

Using this, we see that

$$\frac{d^n}{dt^n} \Big|_{t=0} \frac{1}{(1-t\tau)^{\frac{1}{2}}} e^{\frac{1}{\tau(1-t\tau)}w^2} e^{-\frac{1}{\tau}w^2} = \left(\tau^{-n} \frac{d^n}{dx^n} (x^{\frac{1}{2}+n} e^{\frac{1}{\tau}x}) \right) x^{-\frac{1}{2}} e^{-\frac{1}{\tau}x}.$$

It follows that setting $x = w^2$

$$L_n^{(-\frac{1}{2})}(w^2, \tau) = \frac{1}{n!} \left(\tau^{-n} \frac{d^n}{dx^n} (x^{\frac{1}{2}+n} e^{\frac{1}{\tau}x}) \right) x^{-\frac{1}{2}} e^{-\frac{1}{\tau}x}$$

As in the case of Hermite polynomials, this formula is used to obtain the orthogonality property of $L_n^{(-\frac{1}{2})}(w^2, \tau)$.

Assuming that $\text{Re}\tau < 0$ and restricting $x = w^2$ to the real axis, we want to show that

$$\int_{\mathbb{R}} x^{\frac{1}{2}} e^{\frac{1}{\tau}x} L_n^{(-\frac{1}{2})}(x, \tau) L_m^{(-\frac{1}{2})}(x, \tau) dx = \delta_{n,m}.$$

First remark that $L_n(x, \tau)$ is a polynomial of degree n .

$$\int_{\mathbb{R}} x^{\frac{1}{2}} e^{\frac{1}{\tau}x} L_n^{(-\frac{1}{2})}(x, \tau) L_m^{(-\frac{1}{2})}(x, \tau) dx = \int_{\mathbb{R}} \frac{1}{\tau^n} \frac{1}{n!} \left(\frac{d^n}{dx^n} x^{\frac{1}{2}+n} e^{\frac{1}{\tau}x} \right) L_m^{(-\frac{1}{2})}(x, \tau) dx.$$

If $n \neq m$, one may assume that $n > m$ without loss of generality. Hence, this vanishes by integration by parts n times.

For the case $n = m$, recalling $L_n^{(-\frac{1}{2})}(x, \tau)$ is a polynomial of degree n , and taking $\frac{d^n}{dx^n}$ of both sides of the equality in the next line:

$$L_n^{(-\frac{1}{2})}(x, \tau) = \frac{1}{n!} \frac{d^n}{dt^n} \Big|_{t=0} \frac{1}{(1-t\tau)^{\frac{1}{2}}} e^{\tau \frac{1}{(1-t\tau)}x} e^{-\frac{1}{\tau}x},$$

we have

$$\begin{aligned} \frac{d^n}{dx^n} L_n^{(-\frac{1}{2})}(x, \tau) &= \frac{1}{n!} \frac{d^n}{dt^n} \Big|_{t=0} \frac{d^n}{dx^n} \frac{1}{(1-t\tau)^{\frac{1}{2}}} e^{\frac{t}{1-t\tau}x} \\ &= \frac{1}{n!} \frac{d^n}{dt^n} \Big|_{t=0} \frac{t^n}{(1-t\tau)^{\frac{1}{2}+n}} e^{\frac{t}{1-t\tau}x}. \end{aligned}$$

But the last term does not contain x for this must be degree 0. Hence,

$$\frac{d^n}{dx^n} L_n^{(-\frac{1}{2})}(x, \tau) = \frac{1}{n!} \frac{d^n}{dt^n} \Big|_{t=0} \frac{t^n}{(1-t\tau)^{\frac{1}{2}+n}} = 1.$$

3.5.2 Intertwiners are 2-to-2 mappings

The intertwiner $I_{\tau}^{\tau'}$ is defined by $e^{\frac{1}{4}(\tau'-\tau)\partial_w^2}$. The case of exponential functions of quadratic forms is treated by solving the evolution equation

$$\frac{d}{dt} f_t(w) = \partial_w^2 f(w), \quad f_0(w) = ce^{aw^2}.$$

Setting $f_t = g(t)e^{q(t)w^2}$, this equation becomes

$$\begin{cases} \frac{d}{dt} q(t) = 4q(t)^2 & q(0) = a \\ \frac{d}{dt} g(t) = 2g(t)q(t) & g(0) = c \end{cases}$$

Solving this gives $g(t)e^{q(t)w^2} = \frac{c}{\sqrt{1-4ta}}e^{\frac{a}{1-4ta}w^2}$. Plugging in $t = \frac{1}{4}(\tau' - \tau)$, we obtain

$$I_{\tau}^{\tau'}(ce^{aw^2}) = \frac{c}{\sqrt{1-(\tau'-\tau)a}}e^{\frac{a}{1-(\tau'-\tau)a}w^2}.$$

Note that this also covers the case for intertwiners for $e_{*}^{\frac{t}{i\hbar}\langle \mathbf{a}, \mathbf{u} \rangle}$.

To reveal its double-valued nature, we rewrite this as follows:

$$(3.26) \quad I_{\tau}^{\tau'}\left(\frac{c}{\sqrt{1-\tau t}}e^{\frac{t}{1-\tau t}w^2}\right) = \frac{c}{\sqrt{1-\tau' t}}e^{\frac{t}{1-\tau' t}w^2}.$$

Since the branching singular point of the double-valued section of the source space moves by the intertwiners, $I_{\tau}^{\tau'}$ must be viewed as a 2-to-2 mapping.

To explain the detail, we construct two sheets with slit from τ^{-1} to ∞ , and denote points by $(t; +)_{\tau}$ or $(t; -)_{\tau}$. $I_{\tau}^{\tau'}$ has the property that $I_{\tau}^{\tau'}((t; \pm)_{\tau}) = (t; \pm)_{\tau'}$ as a set-to-set mapping, and one may define this locally a 1-to-1 mapping. Note that

$$I_{\tau}^{\tau''} I_{\tau'}^{\tau''} I_{\tau}^{\tau'}((t, \pm)_{\tau}) = (t, \pm)_{\tau},$$

but this is neither the identity nor -1 . This depends on t discontinuously.

On the other hand, we want to retain the feature of complex one parameter group. For that purpose, we have to set $:e_{*}^{0w^2}:_{\tau} = 1$, the multiplicative unit for every expression. The problem is caused by another sheet, for we have to distinguish 1 and -1 .

It is important to recognize that there is no effective theory to understand such a vague system. This is something like an *air pocket* of the theory of point set topology. This is not a difficult object, but an object that can be treated case by cases to avoid its ambiguous character. However, the important thing is that such phenomena occur very often in the stage of applying calculus. As it will be seen in the next section, this system forms an object which may be viewed as a *double covering group* of \mathbb{C} . Well, this is absurd since \mathbb{C} is simply connected !

Note A toy model of such a strange object is given by considering Hopf fibering $S^3 = \coprod_{x \in S^2} S_x^1$ where S_x^1 is the fiber at $x \in S^2$. Let \tilde{S}_x^1 be the double covering group of each fiber. Then, consider the disjoint sum $\coprod_{x \in S^2} \tilde{S}_x^1$.

3.5.3 Another definition of $*$ -inverses of $a^2 + w_{*}^2$

Since $a^2 + w_{*}^2 = (a + iw) * (a - iw)$, its inverse may be defined by

$$(a + iw)_{*+}^{-1} * (a - iw)_{*-}^{-1} = \frac{1}{2a}((a + iw)_{+}^{-1} + (a - iw)_{-}^{-1}), \quad \text{Re } \tau > 0.$$

Hence, we have for $a \neq 0$,

$$\begin{aligned} \frac{1}{2a}((a+iw)_+^{-1} + (a-iw)_-^{-1}) &= \int_{-\infty}^0 \frac{1}{2a} (e_*^{t(a+iw)} + e_*^{t(a-iw)}) dt \\ &= \int_{-\infty}^0 \frac{1}{a} e^{at} \cos_*(tw) dt, \quad \operatorname{Re} \tau > 0. \end{aligned}$$

On the other hand, using

$$:e_*^{tw^2}:_\tau = \frac{1}{\sqrt{1-t\tau}} e^{\frac{t}{1-t\tau}w^2}, \quad \tau \in \mathbb{C}.$$

and the exponential law, we define another $*$ -inverse of $a^2 + w_*^2$, for $a^2 > 0$ by

$$(a^2 + w_*^2)^{-1} = \int_{-\infty}^0 e_*^{t(a^2 + w_*^2)} dt, \quad \tau \neq 0.$$

In precise,

$$\int_{-\infty}^0 :e_*^{t(a^2 + w_*^2)}:_\tau dt = \int_{-\infty}^0 \frac{e^{a^2 t}}{\sqrt{1-t\tau}} e^{\frac{t}{1-t\tau}w^2} dt, \quad a^2 > 0.$$

The difference of these two inverses satisfies $(a^2 + w_*^2) * f_*(w) = 0$, hence it is given by a linear combination of $\delta_*(w+ia)$ and $\delta_*(w-ia)$.

3.6 Star-exponential functions of higher order

Note that a $*$ -monomial $e^{im\theta} w_*^m$ may be defined by $e^{im\theta} w_*^m = \int e^{im\theta} x^m \delta_*(x-w) dx$, but noting that $e^{im\theta} w_*^m = (e^{i\theta} w)_*^m$, this may be defined as $\int x^m \delta_*(x-e^{i\theta} w) dx$.

Lemma 3.5 *If both $f(x)$ and $f(e^{i\theta} x)$ are tempered distribution, then it holds the equality*

$$\int f(x) \delta_*(x-e^{i\theta} w) dx = \int f(e^{i\theta} x) \delta_*(x-w) dx$$

In particular, it holds the identity

$$\int f(x) \delta_*(x-e^{i\theta} w) dx = \int f(-x) \delta_*(x+e^{i\theta} w) dx$$

Consider the evolution equation

$$\frac{d}{dt} f_t(w) = i w_*^\ell * f_t(w), \quad f_0(w) = 1$$

for star-exponential functions of higher order $\ell \geq 3$. In precise, this is considered under suitable τ -expressions:

$$\frac{d}{dt} :f_t(w):_\tau = i :w_*^\ell :_\tau * :f_t(w):_\tau, \quad :f_0(w):_\tau = 1.$$

This is a partial differential equation of order $\ell \geq 3$, but Proposition 1.2 shows that there is no real analytic solution, if the expression parameter $\tau \neq 0$.

However, as e^{isx^ℓ} is a slowly increasing function of $x \in \mathbb{R}$ for every real number s and for every positive integer ℓ , the formula (3.1) gives that $\int_{\mathbb{R}} e^{isx^\ell} : \delta_*(x-w) :_\tau dx$ for $\operatorname{Re} \tau > 0$ is a one parameter group which may be denoted by $e_*^{isw^\ell}$.

As $e^{ise^{i\theta}x^\ell}$ is never slowly increasing if $e^{i\theta} \notin \mathbb{R}$, the parameter s of $e_*^{isw^\ell} = \int_{\mathbb{R}} e^{isx^\ell} : \delta_*(x-w) :_\tau dx$ cannot be complexified. However, note that

$$(3.27) \quad \int_{\mathbb{R}} e^{isx^\ell} \delta_*(x - e^{i\frac{\theta}{\ell}} w) dx = e_*^{is(e^{i\frac{\theta}{\ell}} w)^\ell}.$$

If r.h.s. is equal to $e_*^{ise^{i\theta}w^\ell}$, then we can complexify the parameter s , but in what follows we see this is not true. Note again that

$$: \delta_*(x - e^{i\frac{\theta}{\ell}(\theta+2\pi)} w) :_\tau \neq : \delta_*(x - e^{i\frac{\theta}{\ell}} w) :_\tau.$$

We define

$$(3.28) \quad I_\ell(s, \theta, \tau) = \int e^{isx^\ell} : \delta_*(x - e^{i\frac{\theta}{\ell}} w) :_\tau dx = \int_{\mathbb{R}} \mathfrak{F}^{-1}(e^{isx^\ell})(t) : e_*^{-ite^{i\frac{\theta}{\ell}} w} :_\tau dt, \quad \operatorname{Re} e^{2\frac{i}{\ell}\theta} \tau > 0.$$

It is useful to keep the equality $I_\ell'(s, \theta, \tau) = I_\ell(s, \theta, \tau')$ in mind.

In this section we show that $I_\ell(s, \theta, \tau)$ is defined on some sector in the universal covering space $\tilde{\mathbb{C}}_*$ of $\mathbb{C} \setminus \{0\}$ and it behaves as if it were an ℓ -covering group of the $*$ -exponential function $e_*^{ise^{i\theta}w^\ell}$.

As $\operatorname{Re} e^{2\frac{i}{\ell}\theta} \tau > 0$, $e^{-\frac{1}{4}e^{2\frac{i}{\ell}\theta}\tau t^2}$ is a rapidly decreasing in t , and so also $e^{ite^{i\frac{\theta}{\ell}} w} e^{-\frac{1}{4}e^{2\frac{i}{\ell}\theta}\tau t^2}$ on every compact domain of w . Thus, this is well-defined for every $s \in \mathbb{R}$. It is easy to see that $I_\ell(0, \theta, \tau) = 1$.

Integration by parts twice gives that

$$(3.29) \quad \partial_s I_\ell(s, \theta, \tau) = i e^{i\theta} w_*^\ell *_\tau I_\ell(s, \theta, \tau), \quad \partial_\theta I_\ell(s, \theta, \tau) = (is) i e^{i\theta} w_*^\ell *_\tau I_\ell(s, \theta, \tau) = is \partial_s I_\ell(s, \theta, \tau).$$

(3.29) gives also that

$$\frac{d^k}{d(e^{i\theta}s)^k} I_\ell(s, \theta, \tau) \Big|_{s=0} = e^{-ik\theta} \partial_s^k I_\ell(s, \theta, \tau) \Big|_{s=0} = : i^k w_*^{k\ell} :_\tau.$$

Thus, the Taylor series of $I_\ell(s, \theta, \tau)$ at $s = 0$ is the diverging series $\sum_k \frac{(ise^{i\theta})^k}{k!} : w_*^{k\ell} :_\tau$ for $\ell \geq 3$ and $\tau \neq 0$.

Lemma 3.6 *If a complex valued smooth function $f(s, \theta)$ satisfies the partial differential equation*

$$is \partial_s f(s, \theta) = \partial_\theta f(s, \theta).$$

Then, $f(s, \theta)$ is a holomorphic function on the open subset

$$\{(s, \theta); s \in \mathbb{R}, \operatorname{Re} e^{2\frac{i}{\ell}\theta} \tau > 0\}$$

of the universal covering space of $\mathbb{C} \setminus \{0\}$.

Proof Rewrite the Cauchy-Riemann equation by the polar coordinate system $se^{i\theta}$. Note that

$$\partial_x = \cos \theta \partial_s - \frac{1}{s} \sin \theta \partial_\theta, \quad \partial_y = \sin \theta \partial_s + \frac{1}{s} \cos \theta \partial_\theta.$$

The condition $is\partial_s f(s, \theta) = \partial_\theta f(s, \theta)$ shows that $f = u(s, \theta) + iv(s, \theta)$ satisfies the Cauchy-Riemann equation on the space $s \neq 0$.

Suppose $\tau = \tau_0 e^{i\sigma}$, $\tau_0 > 0$. Then the defining domain is $|\frac{2}{\ell}\theta + \sigma| < \frac{\pi}{2}$. It is precisely

$$(3.30) \quad D_\sigma = \{(s, \theta, \sigma); s \in \mathbb{R}, -\frac{\ell}{4}\pi - \frac{\ell\sigma}{2} < \theta < \frac{\ell}{4}\pi - \frac{\ell\sigma}{2}\}$$

in the universal covering space of $\mathbb{C} \setminus \{0\}$ depending on σ . □

3.6.1 ℓ -covering property

Consider the defining domain (3.30) of $I_\ell(s, \theta, \tau)$. If ℓ is big enough, say $\ell > 8$, then D_σ for any fixed σ contains an interval $[\theta, \theta + 2\pi]$. Hence, $I_\ell(s, \theta, \tau)$ and $I_\ell(s, \theta + 2\pi, \tau)$ are defined under the same expression parameter. If these are equal, then $I_\ell(s, \theta, \tau) = :e_*^{se^{i\theta}w_*^\ell} :_\tau$ is well-defined on $se^{i\theta} \in \mathbb{C} \setminus \{0\}$, and the origin is a removable singular point. Since this against Proposition 1.2, we have that $I_\ell(s, \theta, \tau) \neq I_\ell(s, \theta + 2\pi, \tau)$.

Using the second identity of Lemma 3.5 one can make the same argument in case that D_σ contains an interval $[\theta, \theta + \pi]$, but if we consider intertwiners together with the joint parallel translations, we can conclude that

$$I_\ell(s, \theta + 2\pi, \tau') \neq I_\tau^{\tau'} I_\ell(s, \theta, \tau).$$

On the other hand, as $I_\ell(s, \theta + 2\pi\ell, \tau) = \int e^{isx^\ell} : \delta_*(x - e_*^{i\frac{\theta+2\pi\ell}{\ell}w}) :_\tau dx = \int e^{isx^\ell} : \delta_*(x - e_*^{i\frac{\theta}{\ell}w}) :_\tau dx$ we have

$$I_\ell(s, \theta + 2\pi\ell, \tau) = I_\ell(s, \theta, \tau).$$

Thus, we conclude the following:

Proposition 3.5 *What is defined by $I_\ell(s, \theta, \tau)$ is not $e_*^{ise^{i\theta}w_*^\ell}$ but its branched ℓ -covering.*

On the other hand, it holds for every k

$$\begin{aligned} \frac{d}{ds} I_\ell(s, \theta + 2\pi k, \tau) &= \int ix^\ell e^{isx^\ell} : \delta_*(x - e_*^{i\frac{\theta+2\pi k}{\ell}w}) :_\tau dx \\ &= \int ix^\ell : \delta_*(x - e_*^{i\frac{\theta+2\pi k}{\ell}w}) :_\tau dx * I_\ell(s, \theta + 2\pi k, \tau). \end{aligned}$$

We saw $I_\ell(s, \theta+2\pi k, \tau) \neq I_\ell(s, \theta, \tau)$, but as w_*^ℓ is a polynomial, we see

$$\int ix^\ell : \delta_*(x - e^{i\frac{\theta+2\pi k}{\ell}} w) :_\tau dx = i(e^{i\frac{1}{\ell}\theta} w_*)^\ell = ie^{i\theta} w_*^\ell$$

It follows for every k

$$\frac{d}{ds} I_\ell(s, \theta+2\pi k, \tau) = :ie^{i\theta} w_*^\ell :_\tau *_\tau I_\ell(s, \theta+2\pi k, \tau), \quad I_\ell(0, \theta+2\pi k, \tau) = 1.$$

Thus, we have the following:

Proposition 3.6 *The uniqueness fails in the evolution equation*

$$\frac{d}{dt} f_t(w) = iw_*^\ell * f_t(w), \quad f_0(w) = 1.$$

There are ℓ -different solutions, which are not real analytic at $t = 0$.

Product structure As $\delta_*(x - e_*^{i\frac{\theta}{\ell}} w) * \delta_*(x' - e_*^{i\frac{\theta'}{\ell}} w) = \delta(x - x') \delta_*(x' - e_*^{i\frac{\theta}{\ell}} w)$, we see

$$\int e^{isx^\ell} \delta_*(x - e_*^{i\frac{\theta}{\ell}} w) dx * \int e^{is'x'^\ell} \delta_*(x' - e_*^{i\frac{\theta'}{\ell}} w) :_\tau dx' = \int e^{i(s+s')x^\ell} \delta_*(x - e_*^{i\frac{\theta}{\ell}} w) dx.$$

However, if $\theta \neq \theta'$, then the product

$$\int e^{isx^\ell} \delta_*(x - e_*^{i\frac{\theta}{\ell}} w) dx * \int e^{is'x'^\ell} \delta_*(x' - e_*^{i\frac{\theta'}{\ell}} w) :_\tau dx'$$

is not defined. It seems that there is no branch point other than the origin $s = 0$.

3.6.2 Star exponential functions of $(\alpha+w)_{*+}^{-1}$

As we have seen already, the $*$ -exponential function $:e_*^{isw}:_\tau$ has high regularity w.r.t. $s \in \mathbb{C}$ when the expression parameters are restricted in a half-space, e.g. $\text{Re } \tau > 0$, and the $*$ -inverse function $(\alpha+w)_{*+}^{-1}$ has high regularity. Thus it seems natural to think that

$$(3.31) \quad e_*^{s(\alpha+w)_{*+}^{-1}} = \sum \frac{s^n}{n!} \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\alpha^{n-1}} (\alpha+w)_{*+}^{-1} = \sum_n \int_{-\infty}^0 \frac{s^n (-it)^{n-1}}{n!(n-1)!} e_*^{it(\alpha+w)} dt$$

is defined with high regularity.

This may be defined by the evolution equation $\frac{d}{ds} f_s = ((\alpha+w)_{*+}^{-1}) * f_s$, $f_0 = 1$. It is natural to think that this is included in the solution of

$$(3.32) \quad (\alpha+w) * \frac{d}{ds} f_s(w) = f_s(w), \quad f_0(w) = 1.$$

But this equation ignores the \pm ambiguities of the inverse.

At first, solving

$$(\alpha+w)*_{\tau}g_{\lambda}(w) = (\alpha+w)g_{\lambda}(w) + \frac{\tau}{2}\partial_w g(w) = \lambda g_{\lambda}(w), \quad \lambda \in \mathbb{C},$$

we see that $g_{\lambda}(w) = c:\delta_*(w+\alpha-\lambda):_{\tau}$, $c \in \mathbb{C}$.

Moreover, if $\text{Im } \lambda < 0$, then

$$\begin{aligned} (w+\alpha)_{*+}^{-1}*\delta_*(w+\alpha-\lambda) &= i \int_{-\infty}^0 ds \int_{-\infty}^{\infty} dt e_*^{is(w+\alpha)} e_*^{it(w+\alpha)} e^{-it\lambda} \\ &= \int_{-\infty}^{\infty} e_*^{i\sigma(w+\alpha-\lambda)} d\sigma i \int_{-\infty}^0 e^{i\lambda s} ds = \frac{1}{\lambda} \delta_*(w+\alpha-\lambda). \end{aligned}$$

It follows that $:\delta_*(w+\alpha-\lambda):_{\tau} e^{\frac{1}{\lambda}s}$ is a real analytic solution of $\frac{d}{ds} h_s(w) = (w+\alpha)_{*+}^{-1} * h_s(w)$ providing $\lambda \neq 0$. Hence, it may be written as

$$:e_*^{s(w+\alpha)_{*+}^{-1}}*\delta_*(w+\alpha-\lambda):_{\tau} = :\delta_*(w+\alpha-\lambda):_{\tau} e^{\frac{1}{\lambda}s}, \quad \text{Re } \tau > 0.$$

But this does not imply the existence of $e_*^{s(w+\alpha)_{*+}^{-1}}$.

To adjust the initial condition, we set $\lambda = \alpha+x-i\eta$, $\eta > 0$, and

$$f_s(w) = \int_{-\infty}^{\infty} e^{s \frac{1}{\alpha+x-i\eta}} :\delta_*(w-x+i\eta):_{\tau} dx.$$

This is well-defined and independent of η by Cauchy's integration theorem whenever $\eta > 0$.

This is the solution of (3.32) satisfying $f_0(w) = 1$ by $\int_{-\infty}^{\infty} :\delta_*(w-x+i\eta):_{\tau} dx = 1$ and this gives also the solution of

$$\frac{d}{ds} \tilde{f}_s(w) = (w+\alpha)_{*+}^{-1} * \tilde{f}_s(w), \quad \tilde{f}_0(w) = 1.$$

Hence,

$$:e_*^{s(w+\alpha)_{*+}^{-1}}:_{\tau} = \int_{-\infty}^{\infty} e^{s \frac{1}{\alpha+x-i\eta}} :\delta_*(w-x+i\eta):_{\tau} dx. \quad s \in \mathbb{C}$$

is a complex one parameter group.

As $\text{Im } \lambda < 0$, we see $\lim_{t \rightarrow -\infty} e^{-\frac{1}{\lambda}it} = 0$, hence

$$\lim_{t \rightarrow \infty} e_*^{-it(w+\alpha)_{*+}^{-1}} = 0, \quad \text{and} \quad i \int_{-\infty}^0 e_*^{it(w+\alpha)_{*+}^{-1}} dt = w+\alpha = \left((w+\alpha)_{*+}^{-1} \right)_{*+}^{-1}.$$

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